

A High-Order Series Solution for Standing Water Waves

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SUMMARY A method of time-dependent conformal mapping is introduced to simplify the power series solution procedure for time and space periodic standing waves. Wave profiles are computed by use of rational fractions. The convergence properties of the series are also discussed.

1 INTRODUCTION

Standing gravity wave problems are more difficult to analyse than are steady wave motions, such as the classical Stokes progressive wave, because of the complications introduced by their time dependence. Nevertheless, approximate small-amplitude expansions have been found for many cases of interest: two- and three-dimensional waves on a fluid of finite and infinite depth; composite waves of more than one fundamental frequency; interfacial waves in multi-layered fluids; and, effects due to surface tension. A survey of these and other standing wave problems may be found in the review articles of Wehausen and Laitone (1960) and Wehausen (1965).

In the present work we take a fresh look at the most fundamental standing wave problem: the two-dimensional, simply-periodic, irrotational motion of a perfect fluid in an infinitely deep and laterally unbounded ocean. In contrast to the progressive wave case, an existence proof for standing waves has not yet been found. In addition, there remain doubts as to the form of the highest standing wave profile.

Finite amplitude deep-water standing waves were first investigated by Rayleigh (1915) who obtained a third-order solution in an assumed small-amplitude expansion. Two important features of deep-water stationary waves become apparent in the first few orders: the maximum elevation of the surface above the mean water level exceeds the maximum depression below it, and the frequency of the wave motion is decreased by an increase in wave amplitude. In the most ambitious effort to date, Penney and Price (1952) carried the perturbation expansion to the fifth order. They found that there is no time during the period of the wave motion when the free surface is perfectly flat (a fourth order effect). Even more surprising, they concluded that the crest of the highest wave has a right-angled nodal form in contrast with that of the greatest stable travelling wave for which the nodal angle is 120° . These predictions were later confirmed experimentally by Taylor (1953) who, while he doubted some of the underlying assumptions of Penney and Price's theoretical analysis, nevertheless believed their results to be correct.

The use of Eulerian coordinates, as employed in the above theoretical studies, involves the application of a nonlinear boundary condition on a free surface whose location is initially unknown. A considerable amount of labour is involved in "transferring"

this boundary condition to the unperturbed fluid level. In a Lagrangian coordinate system, on the other hand, the location of the free surface can be assumed to be known; additional complexity arises from the satisfaction of the continuity and irrotationality conditions, however. In this paper, we will employ a third method, which appears to combine the advantages of the other two schemes. It is a direct adaptation of the conformal mapping method described in Whitney (1971). Here we seek a time-dependent mapping $z(\zeta, t)$ which maps the fluid region in the physical or z -plane onto a prescribed, time-independent, domain in the ζ -plane.

2 MATHEMATICAL FORMULATION

The region of the physical or z -plane occupied by the fluid at each instant of time t is mapped onto the lower half ζ -plane according to the transformation

$$z = x + iy = z(\zeta, t; \varepsilon) \quad (1)$$

where ε is a parameter that subsequently will be identified with the wave height and $\zeta = \xi + i\eta$. The free surface corresponds to $\eta = 0$ (see Figure 1). We introduce the usual complex velocity potential $f = \phi + i\psi$ and the complex velocity

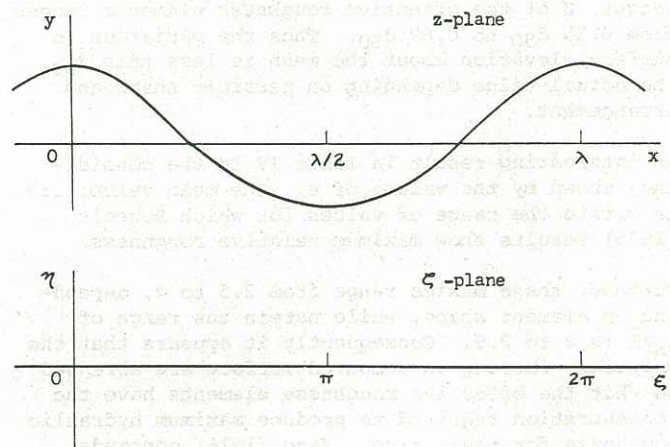


Figure 1 One wave cycle in the physical and transformed planes.

$w = df/dz = u - iv$. Henceforth, upper-case letters will denote the direct function dependence of these

quantities on ζ and t :

$$\begin{aligned} F(\zeta, t) &= \Phi + i\Psi \equiv f(z(\zeta, t), t) \\ W(\zeta, t) &= U - iV \quad w(z(\zeta, t), t) \end{aligned} \quad (2)$$

W , F and z are related by

$$W = \frac{dF}{dz} = \frac{F_{\zeta}}{z_{\zeta}} \quad (3)$$

where subscripts signify partial differentiation.

The kinematic boundary condition states that the normal component of fluid velocity (U, V) of a particle occupying a point on the surface is equal to the normal component of the surface velocity (x_t, y_t) at that point.

Noting that $(-y_{\zeta}, x_{\zeta})$ is a normal vector to the surface, we obtain

$$\text{Im}\{F_{\zeta} - z_{\zeta} \bar{z}_t\} = 0 \quad \text{on } \eta = 0, \quad (4)$$

where use has been made of (3) and the bar signifies complex conjugation.

The pressure p at any point in the water, in the z -plane, is given by the Bernoulli equation

$$\frac{p-p_0}{\rho} = -\phi_t - \frac{1}{2}w\bar{w} - gy \quad (5)$$

Here ρ is the density, g the acceleration of gravity, and p_0 the atmospheric pressure which may, without loss of generality, be set equal to zero. On the free surface, the dynamic boundary condition states that $p = p_0$. Thus

$$\phi_t + \frac{1}{2}w\bar{w} + gy = 0 \quad (6)$$

on the free surface. We require equation (6) in terms of ζ -plane variables evaluated on $\eta = 0$. Only the velocity potential term involves special treatment. From (2) we have

$$F_t = f_t + f_z z_t = f_t + w z_t.$$

By substituting the real part of this expression into (6), the dynamic boundary condition becomes

$$\phi_t + \frac{1}{2}w\bar{w} + gy - \text{Re}\{W z_t\} = 0 \quad \text{on } \eta = 0. \quad (7)$$

Proceeding now to dimensionless variables, we choose as a characteristic length the wavelength λ and as a characteristic time the unknown period T . Let

$$k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T}$$

be the wave number and frequency for the standing wave motion. We define the dimensionless variables

$$\tilde{z} = k z, \quad \tilde{t} = \omega t, \quad \tilde{F} = \frac{k^2}{\omega} F, \quad \tilde{W} = \frac{k}{\omega} W. \quad (8)$$

We make the substitutions indicated by (8) in (3) and the surface conditions (4) and (7). After dropping the tildes, (3) and (4) remain unchanged, while (7) becomes

$$\phi_t + \frac{1}{2}w\bar{w} + S y - \text{Re}\{W z_t\} = 0 \quad \text{on } \eta = 0 \quad (9)$$

where the frequency parameter S is defined as

$$S = \frac{gk}{\omega^2} \quad (10)$$

If the wave number k is considered to be fixed, then the frequency ω , and hence S , will have to be determined as one of the unknowns of the problem.

The dependent functions z , F , and W are required to be analytic in ζ and t in the lower half

plane. The depth of the water is assumed to be infinite; consequently we require

$$z \sim \zeta, \quad W \sim 0, \quad F \sim 0 \quad \text{as } \eta \rightarrow -\infty \quad (11)$$

to ensure that the disturbances vanish far beneath the surface. The analytic and periodic requirements, together with (11), imply that the functions may be represented by Fourier series of the form

$$z = \zeta + i \sum_{p=0}^{\infty} a_p e^{-ip\zeta} \quad (12a)$$

$$W = i \sum_{p=0}^{\infty} b_p e^{-ip\zeta} \quad (12b)$$

$$F = \sum_{p=0}^{\infty} c_p e^{-ip\zeta} \quad (12c)$$

where we have chosen the η -axis to be a line of symmetry.

The coordinate $y(\xi, t)$ is the vertical displacement of a point on the surface. We introduce ϵ as the half-wave height by requiring

$$2\epsilon = y(0, 0) - y(\pi, 0). \quad (13)$$

The required solution has the property that when ϵ is very small, the surface profile is simply periodic in space and time with frequency and wave number both equal to 2π . That is

$$y = \epsilon \cos x \cos t + O(\epsilon^2). \quad (14)$$

Insertion of expansions (12) into (3), (4) and (9) reveals that it is sufficient to assume a Stokes-type expansion; that is

$$a_p = \sum_{n=0}^{\infty} \alpha_{pn} \epsilon^{p+2n} \quad (15a)$$

$$b_p = \sum_{n=0}^{\infty} \beta_{pn} \epsilon^{p+2n} \quad (15b)$$

$$c_p = \sum_{n=0}^{\infty} \gamma_{pn} \epsilon^{p+2n} \quad (15c)$$

and

$$S = 1 + \sum_{n=1}^{\infty} \sigma_n \epsilon^{2n} \quad (15d)$$

where the doubly-subscripted elements are (necessarily periodic) functions of time and σ_n are constants to be determined.

After equating coefficients of like powers of $\epsilon^{-i\zeta}$ and ϵ , a system of recurrence relations is obtained with the usual property that the unknown elements at any stage are determined, in general, by previously computed elements of lower order. A similar set of relations was obtained by Schwartz (1974) in the analogous series solution for the progressive wave. There, however, the relations were purely algebraic while in the present problem the relations can be shown to be of the form

$$\alpha''_{pn} + p \alpha_{pn} = R_{pn} \quad (16)$$

with similar expressions for β_{pn} and γ_{pn} . The right side of (16) is a time-periodic function of lower order coefficients.

In general, a sufficient assumption for the form of the coefficients is

$$\alpha_{pn} = \sum_{\ell=0}^{\left[\frac{p+2n}{2}\right]} \alpha_{pn\ell} \cos(p+2n-2\ell)t. \quad (17)$$

The expansions for the β and γ array elements, compatible with (17), are sine series. Here $[\]$ signifies the "integer-part" function.

consistency is present, can accurately predict the nature and location of a limiting singularity in a series expansion. Such tests were performed for a number of values of ξ with $t = 0$, corresponding to points on the highest wave profile for given ϵ . ϵ^* , in general, is a complex number whose magnitude and argument are functions of ξ . Thus the series (22) is not uniformly convergent. The maximum value of $|\epsilon^*|$ for which the series converges for any value of ξ is 0.30 corresponding to a wave height/length ratio of 0.095. This is surely not the value of $|\epsilon^*|$ for the highest possible wave but it does indicate that a method of analytic continuation is required to "sum" the series. For $\xi = 0, \epsilon^*$ is a real number; thus this value of ϵ^* should correspond to the highest possible wave. Although this ϵ^* cannot be estimated especially accurately, it is almost certainly in the range 0.6 to 0.7, corresponding to wave-height to length ratios between 0.19 and 0.22. The experiments of Taylor (1953) yielded values for this number between 0.22 and 0.24.

A simple method of analytic continuation which is appropriate here involves recasting the original series (22) in the form of rational fractions (Padé Approximants). While the original series will only converge within a circle of radius $|\epsilon^*|$, the sequence of rational fractions may be expected to converge in a much larger domain whose extent is limited primarily by the location of the non-polar singularities of $z(\epsilon)$. Here we use only those rational fractions where the numerator and denominator are of the same order (diagonal Padé Approximants). From the original $O(\epsilon^{25})$ solution, therefore, we can obtain a sequence of 13 fractions, for any values of ξ , t , and ϵ corresponding to ratios of polynomials with order increasing from 0 to 12. These new sequences will usually converge rapidly if $|\epsilon^* - \epsilon|$ is not very small.

Figure 3 shows three wave profiles computed for $t = 0$ and values of ϵ equal to 0.4, 0.5 and 0.6. The ϵ -values for each profile are sufficiently high that straightforward summing of the series would fail in each case. The sequence of Padé Approximants on the other hand converges to at least 5 places for $\epsilon = 0.4$, 4 places for $\epsilon = 0.5$, and about 3 places for $\epsilon = 0.6$. For $\epsilon = 0.65$ (not shown), the convergence is poor and for still larger values of ϵ , the profiles could not be considered to have converged at all. The $\epsilon = 0.6$ profile has a maximum slope of 32.2° at $x = 0.33$. It is interesting to note that the slope does not decrease monotonically from that point but has a second (relative) maximum of 32.0° at $x = 1.0$. The maximum slope for the $\epsilon = .65$ profile can be estimated to be between 43° and 45° and occurs quite near the crest. It seems quite likely that the highest standing wave has a sharp crest with a 90° included angle, as predicted by Penney and Price. We suspect, however, that the maximum profile slope may slightly exceed 45° . A similar effect for the progressive wave has recently been revealed by the meticulous calculations of Longuet-Higgins and Fox (1977).

Figure 4 shows the free-surface shape for $\epsilon = 0.60$ at various times from $t = 0$ to $\pi/2$, the quarter period value. Notice that the surface is not flat at $t = \pi/2$, the maximum displacement being about 0.004λ at $x = \pi/2$. For $\pi/2 < t < \pi$, the profiles

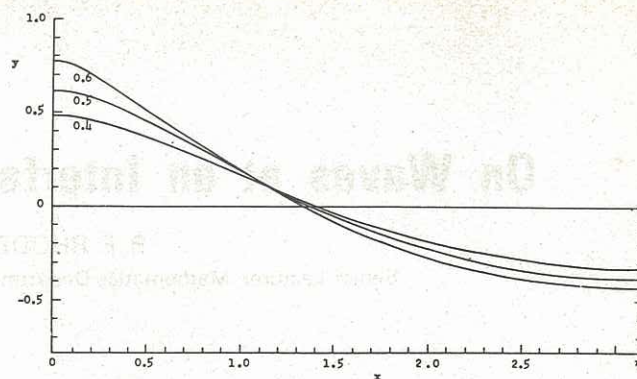


Figure 3 Maximum wave profiles for 3 values of ϵ

will be left-right reflections of those shown and for $\pi < t < 2\pi$, the surface will merely assume the profiles shown, but in reverse order. The shape of the nearly-flat profile, at $t = \pi/2$, is not well represented by equation (21) which is not surprising in view of the relatively large value of ϵ .

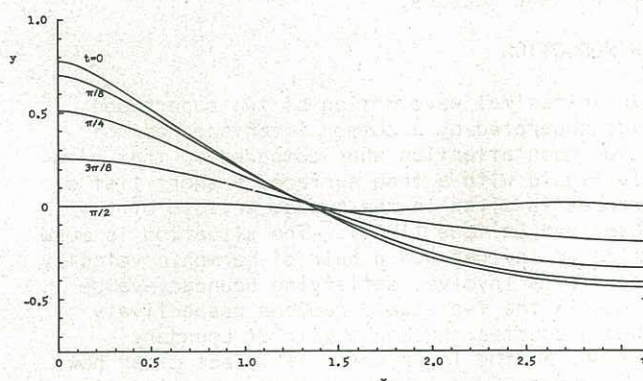


Figure 4 Surface profiles at various times, $\epsilon = 0.6$

Penney and Price (1952) argue that the deceleration at the crest cannot exceed g in magnitude. This conclusion is undoubtedly correct and can be rigorously established without appeal to physical intuition. In terms of ζ -plane variables, this condition yields the inequality

$$V_t + S \geq 0 \quad \text{at } t = 0 \text{ and } \zeta = 0. \quad (23)$$

By writing the left side of (23) as a series in ϵ , recasting as a sequence of rational fractions, and seeking the value of ϵ for which equality is satisfied, we obtain $\epsilon_{\max} \approx 0.67$. This number is consistent with our earlier estimates.

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