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Reflection and Transmission of Water Waves by a Submerged Shelf of Prescribed Shape

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SUMMARY The paper presents a numerical solution to the problem of reflection of waves, in water of finite depth, by a submerged shelf. Using linearized water-wave theory, the problem is reduced to that of solving coupled integral equations in the boundary values of the velocity potential and its normal derivative. Straightforward numerical techniques are used to solve the problem. The computer programme has as input the co-ordinates of the prescribed bottom topography and outputs the reflection and transmission coefficients as functions of frequency.

1 INTRODUCTION

This paper presents a method for determining the amount of reflection that takes place when a plane progressive water wave moves from one region of constant depth to another region of the same or different constant depth, separated by an intermediate region of given bottom topography. As in most previous work only the two dimensional problem is considered, although extension to three dimensions is possible.

It is assumed that wave amplitudes are small so that the problem may be linearized. The fluid is further assumed non-viscous and the flow irrotational, so that Laplace's equation holds throughout the fluid.

The simpler problem of reflection by a vertical step was considered by several authors, including Hilaly (1967), and Bartholomeusz (1958). However, few results have been obtained for the more general problem outlined above.

Evans (1972), using a Green's theorem approach was able to reduce the problem to that of solving one singular integral equation. However, the kernel required in this formulation is extremely complicated and Evans gave no results. For general bottom topography this method appears limited in practical application.

Fitzgerald (1976) has recently solved the problem numerically for certain types of bottom by an inverse method. The results he presents are rather restricted, however, and only those types of bottom which can be mapped uniformly by a specific analytic function to a bottom of fixed depth are considered.

Yeung (1975) by contrast, developed an integral equation which uses a distribution of simple sources. The drawback of this method, however, is that equations of large order must be inverted to obtain accurate results.

This paper steers a mid-course between the two extremes advocated by Yeung and Evans. By using the finite depth Green's function (Wehausen & Laitone, 1960), the number of points at which the potential function and its derivatives must be evaluated is much less than in Yeung's method, yet the kernel functions for the integral equation do not become computationally unmanageable as in the method proposed by Evans.

2 MATHEMATICAL FORMULATION

Cartesian coordinates x and y are used. The region $(-L,0)$ of varying depth is given by $y=-h(x)$ where $h(x)$ is known (see Fig.1). For $x > 0$ we assume constant depth $y = -h_1$, and for $x < -L$ we have $y = -h_2$, where for definiteness we assume $h_2 \geqslant h_1$. An important parameter here is α, given by h_1/h_2.

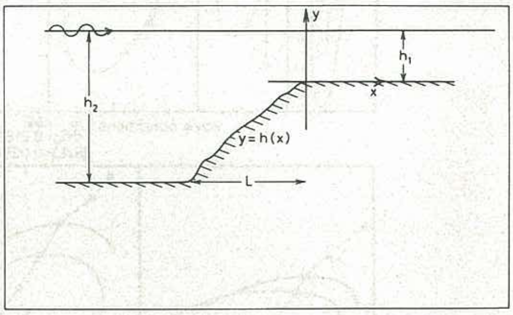

Figure 1

We assume that the fluid motion may be described by the complex velocity potential $\phi(x,y)$, where a sinusoidal time dependence has been suppressed. This potential must satisfy a free-surface condition,

$$\frac{\partial\phi}{\partial y} - \nu\phi = 0, \quad y = 0 \qquad (1)$$

where $\nu = \sigma^2/g$, σ is wave frequency and g the acceleration due to gravity. It must also satisfy the condition that there should be no normal fluid velocity on the bottom, so that

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{for} \quad \begin{array}{l} y = -h_2, -\infty < x \leqslant -L \\ y = -h(x), -L \leqslant x \leqslant 0 \\ \text{and} \quad y = -h_1, 0 < x < \infty \quad . \end{array} \qquad (2)$$

Plane progressive waves of unit amplitude are incident from $x = -\infty$ so that ϕ takes the form

$$\phi \to (e^{iK_2x} + \rho e^{-iK_2x})\frac{\cosh K_2(y+h_2)}{\cosh K_2h_2} \quad \text{as } x\to-\infty \qquad (3)$$

and

$$\phi \to \tau\, e^{iK_1x}\frac{\cosh K_1(y+h_1)}{\cosh K_1h_1} \quad \text{as } x \to \infty \qquad (4)$$

where ρ and τ are the complex-valued reflection and transmission coefficients. K_1 and K_2 are the characteristic wave numbers for waves of frequency σ in water depths h_1, h_2 respectively given by $K_j \tanh K_j h_j = \nu$, $j = 1,2$.

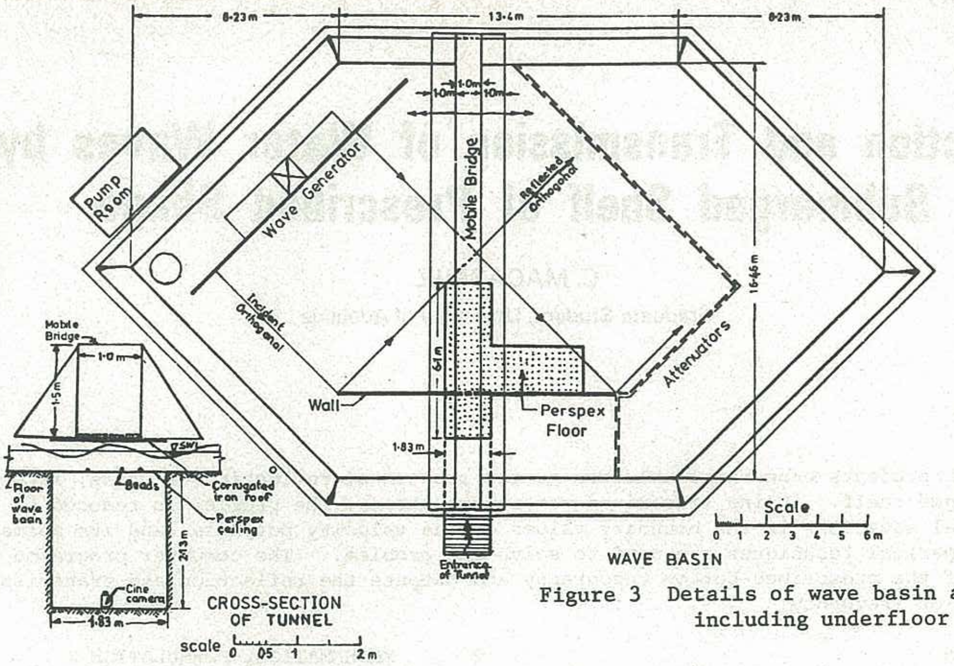

Figure 3 Details of wave basin and appurtenances, including underfloor tunnel

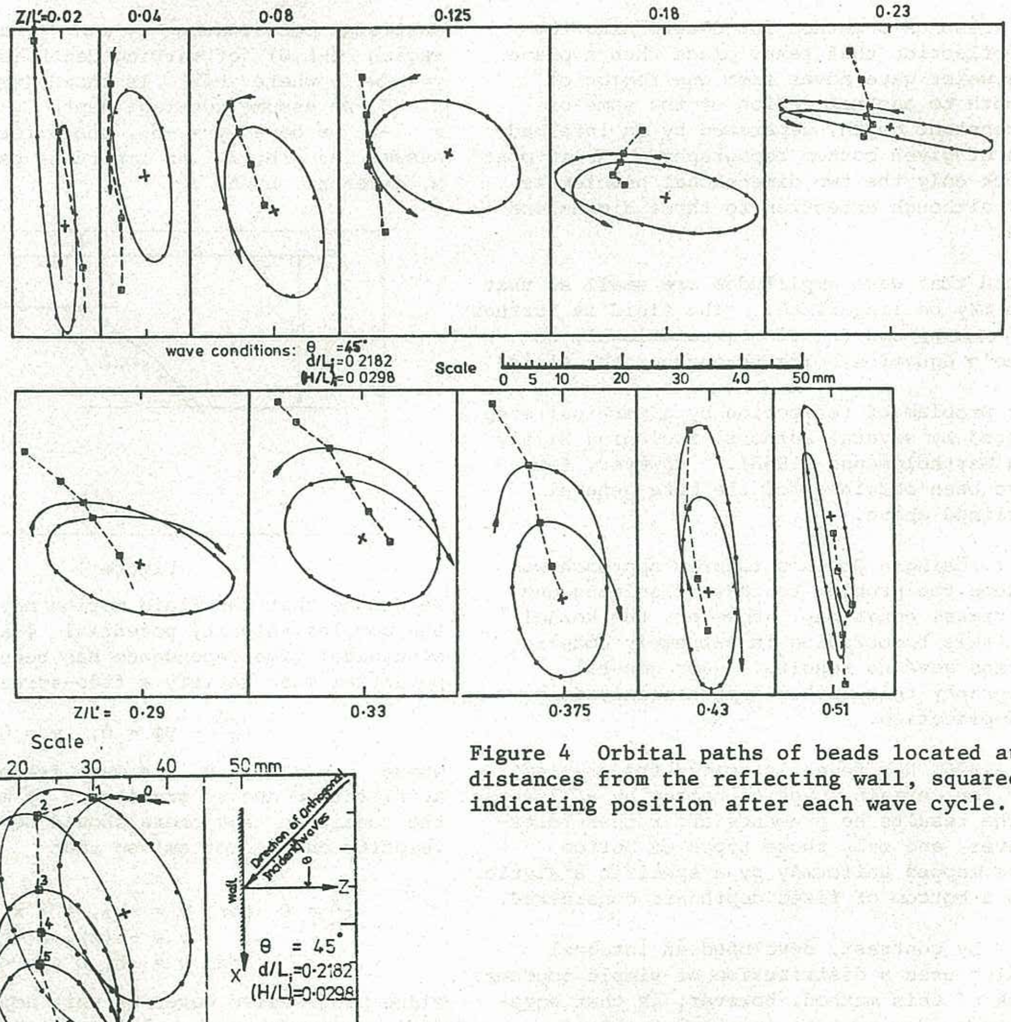

Figure 4 Orbital paths of beads located at specified distances from the reflecting wall, squared points indicating position after each wave cycle.

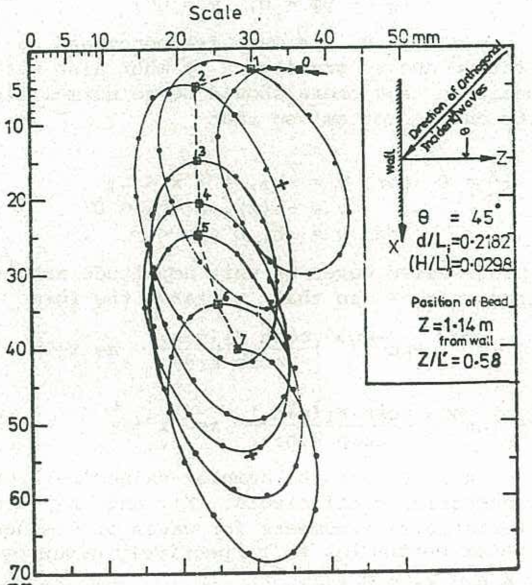

Figure 5 Series of seven orbits recorded beyond Z/L' = ½ from the wall, illustrating influence of incident wave at commencement

3 FORMULATION OF THE INTEGRAL EQUATION

Ideally one might wish to formulate the problem in terms of a single integral equation involving only the velocity potential on $y = -h(x)$ as was done by Evans (1972). As has been mentioned before, however, this leads to an integral equation with an extremely complicated kernel, since the Green's function required must satisfy two radiation conditions for $x \to \pm\infty$, that involve two different wave numbers, as well as Laplace's equation and the boundary conditions (1) and (2). The use of this kernel can be avoided, at the expense of producing more complicated integral equations, if the fluid region is considered to be divided into two parts by the arc $-h_1 \leqslant y \leqslant 0$, $x = 0$.

Application of Green's theorem separately for $x \geqslant 0$ and $x \leqslant 0$ yields two coupled integral equations. For $x \geqslant 0$,

$$\phi(\xi,\eta) = \int_{-h_1}^{0} G_1(x,y)\frac{\partial\phi}{\partial x}(0,y) - \phi(0,y)\frac{\partial G_1}{\partial x}(0,y)\,dy \tag{6}$$

where $G_1(x,y) = G(x,y,\xi,\eta,h_1)$ satisfies (2),(3) and Laplace's equation and behaves like an outgoing source, in water of depth h_1, at $x = \pm\infty$. For $x \leqslant 0$

$$\tfrac{1}{2}\phi(\xi,\eta) = \phi_0(\xi,\eta) + \fint_B \phi(x,y)\frac{\partial G_2}{\partial n}(x,y)\,d\ell - \int_{-h_1}^{0} \frac{\partial\phi}{\partial x}(0,y)G_2(0,y)\,dy + \int_{-h_1}^{0} \frac{\partial G_2}{\partial x}(0,y)\phi(0,y)\,dy \tag{7}$$

where the bar on the integral indicates the Cauchy Principal Value and

$$\phi_0(\xi,\eta) = \frac{\cosh K_2(\eta+h_2)}{\cosh K_2 h_2}\, e^{iK_2\xi} \quad . \tag{8}$$

B is the arc $y = h(x)$, $-L \leqslant x \leqslant 0$.

This set of equations may be solved numerically, as described in the next section, to obtain the boundary values of ϕ and ϕ_x. Once this has been done the determination of the reflection and transmission coefficients, ρ and τ, is straightforward. To determine ρ, for instance, one inserts the calculated values of ϕ and ϕ_x in the limiting form, as $\xi \to -\infty$, of equation (7). Comparison with (3) allows the reflection coefficient to be readily computed. Similar methods may be used to determine the transmission coefficient.

4 NUMERICAL ANALYSIS

To solve (6) and (7) numerically requires some method of discretization. The assumption made to allow this is that both ϕ and ϕ_x are slowly varying on the arc B and on $0 > y > -h_1, x = 0$. Therefore the arc B is divided into N segments (x_j, x_{j+1}), $j = 1,\ldots,N$, such that $x_j < x < x_{j+1}$. In each segment the approximation $\phi(x,y) = \phi_j$ is made. In the same way the arc $0 > y > -h_1, x = 0$ may be divided in to N segments (y_j, y_{j+1}) with $\phi(0,y) = \phi_j$ and $\phi_x(0,y) = \phi_{xj}$ for $j=N+1,\ldots,2N$. The integral equations must be satisfied at the 2N points

$$\xi_i = \frac{(x_i + x_{i+1})}{2} \qquad i=1,\ldots,N \tag{9}$$

and

$$\eta_i = \frac{(y_i + y_{i+1})}{2} \quad ,i=N+1,\ldots,2N \tag{10}$$

In this way, for instance, (6) becomes (with $\xi=0$)

$$\phi_i = \sum_{j=N+1}^{2N} \phi_{xj} F_{ij} \tag{11}$$

where

$$F_{ij} = 2\int_{y_j}^{y_{j+1}} G_1(0,y;0,\eta_i)\,dy. \tag{12}$$

Note that no assumption has been made about the behaviour of the kernel functions in the above approximations. All integrals in the formulation are determined analytically.

Equations (6) and (7) may now be recast in the form,

$$Z\underset{\sim}{\phi} = \underset{\sim}{\phi}_0 \tag{13}$$

where Z is a matrix given by,

$$Z = \begin{bmatrix} \tfrac{1}{2}I-A & B & C \\ \tfrac{1}{2}I-D & E & 0 \\ 0 & I-F & 0 \end{bmatrix} \tag{14}$$

with $A = [A_{ij}]$, $B = [B_{ij}]$ and so on. $A_{ij}, B_{ij},\ldots, F_{ij}$ are of a similar form to (12), I represents the identity matrix and all submatrices are of order N.

The vectors $\underset{\sim}{\phi}$ and $\underset{\sim}{\phi}_0$ are given by,

$$\underset{\sim}{\phi} = [\phi_1,\ldots,\phi_N,\phi_{x1},\ldots,\phi_{xN},\phi_{N+1},\ldots,\phi_{2N}]^t \tag{15}$$

and

$$\underset{\sim}{\phi}_0 = [\phi_{0_1},\ldots,\phi_{0_{2N}},0,\ldots,0]^t \ , \tag{16}$$

with ϕ_{0_1} etc. being the obvious discretization of $\phi_0(\xi,\eta)$.

Solution of (14) by standard inversion techniques therefore solves the original problem.

5 RESULTS

The above was coded on the Adelaide University CDC 6400 computer. It was found that satisfactory accuracy was achieved at moderate frequency by the inversion of a 60×60 matrix Z. However, at higher frequency to maintain the level of accuracy larger matrix inversions were required, and in fact the accuracy obtainable will decrease as frequency increases.

The most obvious problem to test the completed programme on was the simple step, for which results for a wide range of frequency and the depth ratio α have been given by Hilaly (1967). As can be seen in Figure 2 good agreement is obtained, except at high frequency.

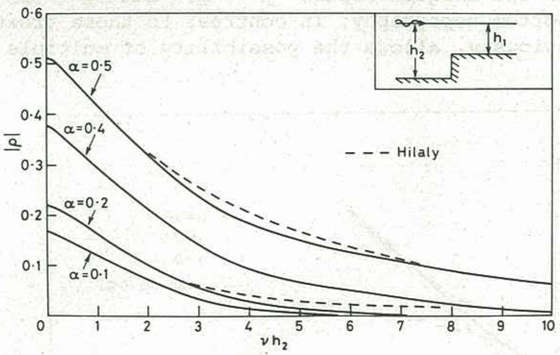

Figure 2

For low frequency, Tuck (1976) has extended the shallow water approximation for this type of problem. His first-order correction indicates that the magnitude of the reflection coefficient is constant for

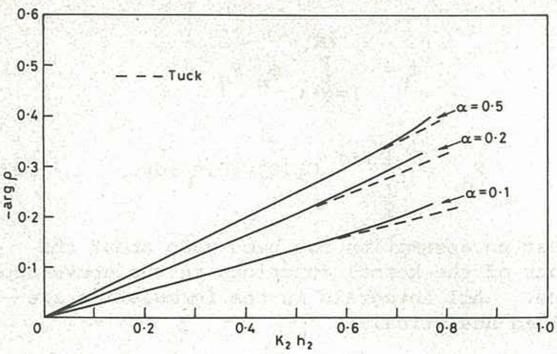

Figure 3

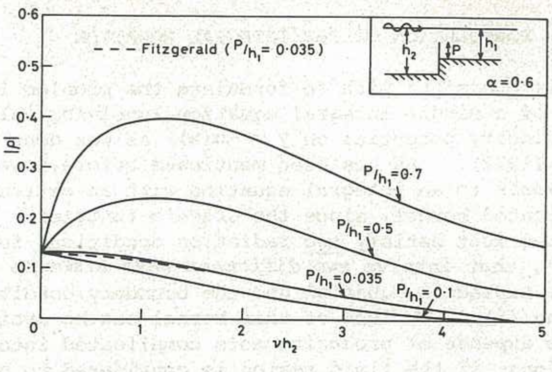

Figure 5

all frequencies but also predicts a linear increase with wave number in the phase of the reflection coefficient. Figure 3 indicates good agreement between this theory and the present work.

At high frequency, the asymptotic form of the reflection coefficient is of interest. Fitzgerald (1976) has developed an analytic theory for some bottom contours (including the simple step) but his final answer is rather obscure. The present programme was used to attempt to find, numerically, the high frequency behaviour for a simple step. As can be seen in Figure 4 it appears that

$$\rho \to e^{-2K_2 h_2 \alpha} \tag{17}$$

No such limit appears in Hilaly's results. Since the result (17) might have been predicted by purely intuitive arguments, it tends to suggest that the present results may be more accurate than Hilaly's for large frequency.

An extension to the simple problem of the step is that of a step with a vertical barrier at $x = 0$ extending part of the way to the free surface. The results for this problem (Figure 5) are interesting in that for fixed α, as the flow is closed off more and more (i.e. as the top of the barrier gets closer to the free surface) a distinct peak in the reflection coefficient becomes apparent, whereas for the simple step the reflection coefficient decreases monotonically with frequency. Fitzgerald (1976) examined this problem but only for small values of p/h. It may also be noticed that however much the gap above the avrrier is closed off, the zero frequency result is always the same, indicating that the transmission of very long waves is independent of barrier geometry.

Another problem of practical interest is that where $y = -h(x)$ is given by a straight line joining the two unequal depths $y = -h_1$ and $y = -h_2$. This bottom topography, in contrast to those treated previously, allows the possibility of multiple maxima and minima in the reflection coefficient, due to the presence of the two corners. It may be seen (Figure 6) that successive minima are indeed obtained as frequency increases. The locations of these minima are quite accurately predicted by shallow water theory and as L/h_2 becomes larger there is agreement in the magnitude of the reflection coefficient. An asymptotic analysis of this problem, for high frequency, would be complicated by the fact that the reflection coefficient does not decay monotonically.

Although the programme was not originally designed for problems where $h_1 = h_2$ these too can be investigated by suitable location of the mesh points. For instance, a single vertical barrier at $x = 0$ extending from $y = -h_2$ partway to the surface may be considered. Results for this problem have been obtained both by Fitzgerald (1976) and by Mei and Black (1969), who used a variational approach. The present work (see Figure 7) shows better agreement with the theory of Mei and Black than with that of Fitzgerald, although the differences between all three solutions are marginal and may be considered to be due to numerical error.

Mei and Black also give results for a square hump on a bottom of fixed finite depth (see Figure 8). Again, good agreement is shown with the present

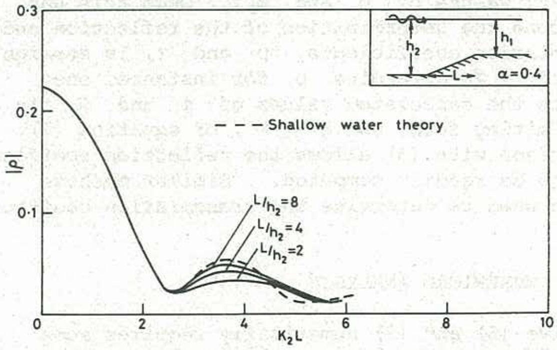

Figure 6

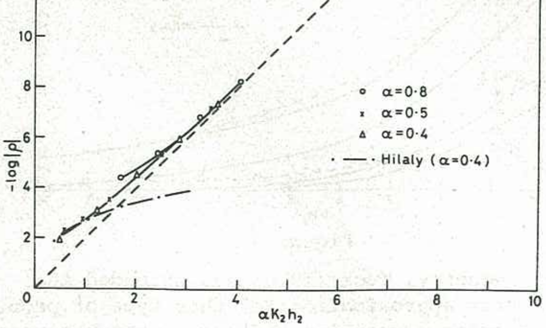

Figure 4

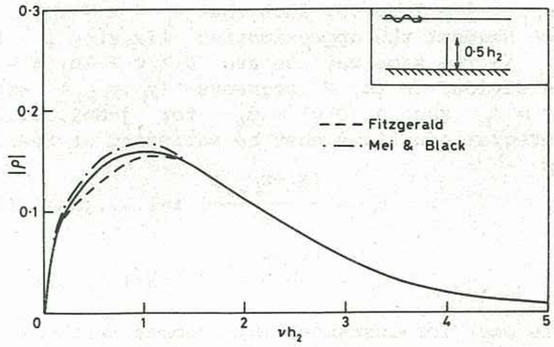

Figure 7

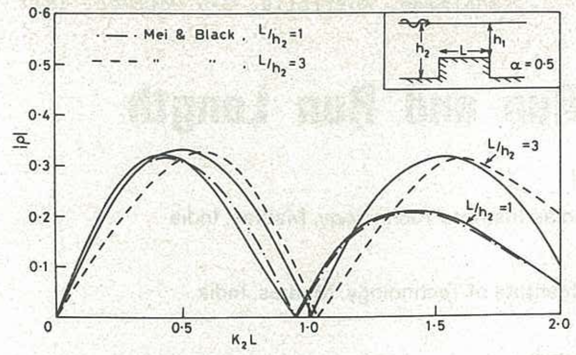

Figure 8

work. Indeed the curve for $L/h_2 = 3$ is especially encouraging since it shows that the programme is able to deal successfully with situations where the assumption made about ϕ being slowly varying on the arc B might be suspect.

6 CONCLUSIONS

The present method shows good agreement with previous work done in this field. It is straightforward to implement and answers can be obtained relatively quickly and efficiently, especially at moderate frequency, where most practical applications would lie.

It has immediate real application in the prediction of reflection of waves over continental shelves and in determining the behaviour of certain types of breakwater. It has advantages over other techniques, such as Fitzgerald's (1976), in that any bottom geometry $y = -h(x)$ may be treated, irrespective of whether it may be described analytically. Where an approximately two-dimensional situation exists, it may be used to approximate wave behaviour once the bottom topography has been determined.

Finally the method is capable of extension (admittedly with some difficulty) to the three-dimensional problem. This would allow non-normal incidence of waves to be taken into account, as well as allowing for more realistic bottom geometry.

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