6th Australasian Hydraulics and
Fluid Mechanics Conference
Adelaide, Australia, 5-9 December 1977

Modified Solution of Unsteady State Cavity Well Hydraulics

C. R. KATAKWAR
Associate Professor, Jawaharlal Nehru Krishi Vishwa Vidyalaya, Jabalpur, India

SUMMARY An unsteady state solution for cavity well with segemental cavity is obtained in term of error and exponential functions employing new boundary condition. The solution can be used for determining aquifer properties and for prediction of steady state discharge for any drawdown.

1 INTRODUCTION

The hydraulics of steady flow to a non penetrating well having hemispherical bottom is given by Muskat (1). Mishra et al (2) and Chauhan (3) have analysed on similar lines to that of Muskat. They have modified the hemispherical bottom considered by Muskat to that of segement of sphere, as expected to be existing in a cavity well. Sarkar (4) has recently studied the unsteady flow to cavity well with hemispherical bottom. In this paper a new boundary condition has been introduced. The well known Laplace transform technique has been used to obtain the solution in form of rapidly convergent error functions. The method of contour integration is employed to obtain the inverse of Laplace transform.

2 NOTATION

K Hydraulic Conductivity

S_s Specific Storage Coefficient

S_o Steady State Drawdown

Q Pumping Rate

r_w Radius of Segement

T Depth of Segement

P Parameter of Laplace Transform

β A new Parameter

$I_{\frac{1}{2}}(\lambda r)$ Bessel Function of First kind of Half order

$K_{\frac{1}{2}}(\lambda r)$ Bessel Function of Second kind of Half order

erf(x) Error Function $= \frac{2}{\sqrt{\pi}} \int_o^x e^{-\eta^2} d\eta$

3 THEORY AND SOLUTION

3.1 Differential Equation and Boundary Conditions

The differential equation governing the symmetrical flow with respect to θ and ϕ in spherical co-ordinate in term of drawdown 's' is given by (Fig.1).

$$\frac{\partial^2 s}{\partial r^2} + \frac{2}{r}\frac{\partial s}{\partial r} = \frac{1}{c}\frac{\partial s}{\partial t} \qquad (1)$$

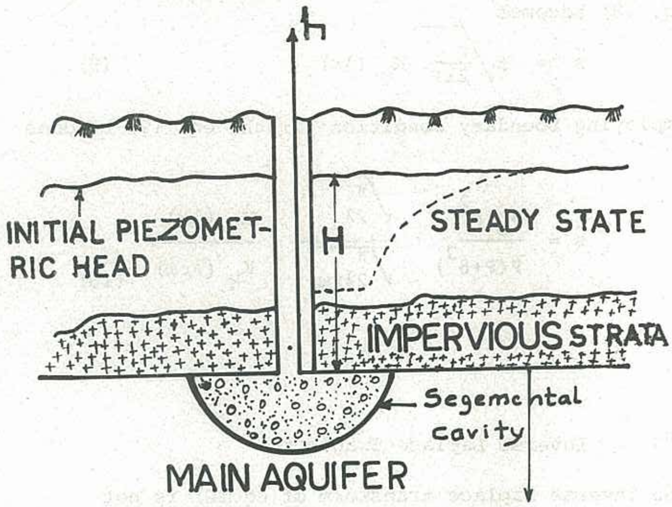

Figure 1 Diagram showing cavity well and coordinate system

where $C = \frac{K}{Ss}$

For the system, time is considered zero just before the start of pumping. Thus at initial time t=o, the whole system is in equilibrium and

$$s(r,o) = o \text{ for } t = o \qquad (2)$$

Second boundary condition is that

$$\text{as } r \to \infty,\ S \to o \text{ i.e.}$$

$$S(\infty, t) = o \text{ for } t > o \qquad (3)$$

Third boundary condition suggested is

$$s(r_w, t) = S_o(1-e^{-\beta^2 t}) \text{ for } t>o \qquad (4)$$

3.2 General Solution

Employing Laplace transform with respect to time 't' and using initial boundary condition (2) the eq (1) transforms into

$$\frac{d^2\bar{s}}{dr^2} + \frac{2}{r}\frac{d\bar{s}}{dr} = \frac{P}{C}\bar{s} \qquad (5)$$

and boundary conditions (3) and (4) are transformed into

$$\bar{s}\,(\infty, P) = o \text{ for } P > o \qquad (6)$$

$$\bar{s}\,(r_w, P) = \frac{S_o\beta^2}{P(P+\beta^2)} \text{ for } P > o \qquad (7)$$

The solution of differential eq. (5) is given in term of modified spherical Bessel functions of first kind and ½ order and second kind and ½ order. A linear combination of these make the complete solution.

$$\bar{s} = A\sqrt{\frac{\pi}{2\lambda r}}\; I_{\frac{1}{2}}\,(\lambda r) + B\sqrt{\frac{\pi}{2\lambda r}}\; K_{\frac{1}{2}}\,(\lambda r) \qquad (8)$$

where $\lambda^2 = P/C$.

Employing boundary condition (6), we get A = o and eq. (8) becomes

$$\bar{s} = \beta\sqrt{\frac{\pi}{2\lambda r}}\; K_{\frac{1}{2}}\,(\lambda r) \qquad (9)$$

Employing boundary condition (7) the eq. (9) becomes

$$\bar{s} = \frac{S_o\beta^2}{P(P+\beta^2)}\;\frac{\sqrt{\frac{\pi}{2\lambda r}}}{\sqrt{\frac{\pi}{2\lambda r\omega}}}\;\frac{K_{\frac{1}{2}}\,(\lambda r)}{K_{\frac{1}{2}}\,(\lambda r\omega)} \qquad (10)$$

3.3 Inverse Laplace Transform

The inverse Laplace transform of eq.(10) is not available, as such using asymptotic expansion of $I\upsilon(z)$ and $K\upsilon(z)$ as given by McLachlan (5) we get

$$\bar{s} = \frac{S_o\beta^2}{P(P+\beta^2)}\;\frac{r_\omega}{r}\; e^{-d\sqrt{P}} \qquad (11)$$

where $d = \dfrac{r - r_\omega}{\sqrt{C}}$

Resolving eq. (11) into partial fractions, we have

$$\bar{s} = \frac{S_o r_\omega}{r}\left(\frac{e^{-d\sqrt{P}}}{P} - \frac{e^{-d\sqrt{P}}}{P+\beta^2}\right) \qquad (12)$$

The inverse Laplace transform of first factor of the bracket is given in table of Laplace transform 6 as

$$L^{-1}\;\frac{e^{-d\sqrt{P}}}{P} = 1 - \mathrm{erf}\left(\frac{d}{2\sqrt{t}}\right) \qquad (13)$$

3.3.1 Inverse integral transform

The inverse Laplace transform of the second factor of the bracket is obtained by the inversion integral along the line x = r (Fig.2)

$$F(t) = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{r-i\beta}^{r+i\beta} e^{Pt}\; e^{-d\sqrt{P}}\;\frac{dP}{P+\beta^2} \qquad (14)$$

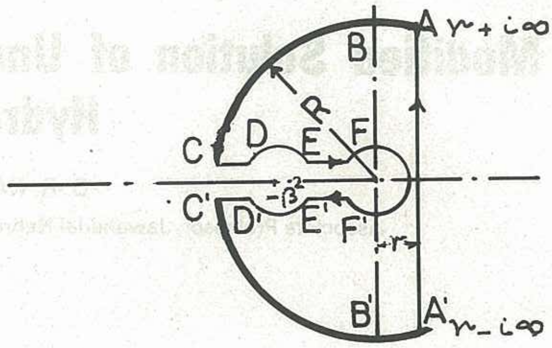

Figure 2 Contour in complex plane 'P'

The sum of contour integral along the path consisting of circular arcs and line segements is zero since the integral is analytic except on the negative real axis. The integral can be written as

$$-\lim_{\beta\to\infty}\int_{r-i\beta}^{r+i\beta} e^{Pt}\, f(P)\,dP = I_{AC} + I_{CD} + I_{DE} + I_{EF} + I_{FF'} + I_{F'E'} + I_{E'D'} + I_{D'C'} + I_{C'A'} \qquad (15)$$

In the integral, on the small circle $r=r_o$, $P=r_o e^{i\theta}$, on the upper semicircle $P=ae^{i\phi}+\beta^2 e^{\pi i}$ and on the lower semicircle $P=ae^{-i\phi}+\beta^2 e^{-\pi i}$ along the upper side of the cut $P = re^{\pi i}$ and on the lower side of the cut $P = re^{-\pi i}$. The integral on the outer circle vanishes as $R\to\infty$.

Upon integration, we get

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{Pt} e^{-d\sqrt{P}}\;\frac{dP}{P+\beta^2} = e^{-\beta^2 t}\cos\beta d + \frac{2}{\pi}\int_0^\infty e^{-\mu^2 t}\;\frac{\cos\mu d}{\beta^2-\mu^2}\;\mu d\mu \qquad (16)$$

And finally we have, inverse transform as

$$S(r,t) = \frac{S_o r_w}{r}\left[\,1 - \mathrm{erf}\left(\frac{d}{2\sqrt{t}}\right) - e^{-\beta^2 t}\cos\beta d - I\,\right] \qquad (17)$$

where

$$I = \frac{2}{\pi}\int_0^\infty \frac{e^{-\mu^2 t}\sin\mu d}{\beta^2-\mu^2}\;\mu d\mu$$

The approximate solution of integral I for large values of t is evaluated (Appendix I) and we get

$$S(r,t) = \frac{S_o r_w}{r}\left[1 - \mathrm{erf}\left(\frac{d}{2\sqrt{t}}\right)(1-\cos\beta d) - \cos\beta d\; e^{-\beta^2 t} - \frac{2\beta t^{\frac{1}{2}}}{\sqrt{\pi}}\sin\beta d\left(e^{-d^2/4t} - 1\right)\right] \qquad (18)$$

The steady state solution for finite radius of influence circle R is given by Chauhan (2) as

$$Q = \frac{2\pi\ TK}{1 - \frac{r_w}{K}} (H - h_c) \qquad (19)$$

Setting R as ∞ and $H-h_c = S_o$ in eq(19), we have

$$S_o = \frac{Q}{2\pi\ TK} \qquad (20)$$

Substituting eq.(20) in eq.(18)

$$S(r,t) = \frac{Q\,r_\omega}{2\pi TKr} \left[1-\mathrm{erf}\left(\frac{d}{2\sqrt{t}}\right)(1-\cos\beta d) - \cos\beta d\ e^{-\beta^2 t} - \frac{2\beta t^{\frac{1}{2}}}{\sqrt{\pi}} \sin\beta d\left(e^{-d^2/4t}-1\right) \right] \qquad (21)$$

4 CONCLUSIONS

(1) Eq.21 can be used for prediction of drawdown for any rate of pumping at any radial distance r after elapse of time t from the start of pumping.

(2) Eq.21 can be used for determining the aquifer properties like hydraulic conductivity and storage coefficient of aquifer and parameter by pumping test.

5 ACKNOWLEDGEMENT

I wish to express my sincere thanks to Prof.S.V. Arya, Dean, Faculty of Agricultural Engineering, J.N.K.V.V., Jabalpur M.P. (India) for his encouragement.

6 REFERENCES

1. MUSKAT, M. (1937). The Flow of Homogeneous Fluids Through Porous Media. McGraw Hill Book Co., New York.

2. MISHRA, A.P., ANJANEYULA, B., and LAL, R. (1970), Design of Cavity Wells, Journal ISAE (India) Vol.II, No.2, pp.12-15.

3. CHAUHAN, H.S. (1971). Steady Flow to a non penetrating cavity well in an artesian aquifer. Journal ISAE (India), Vol. III, No.1, pp.23-27.

4. SARKAR, T.K. (1975). Exact Solution of unsteady cavity well hydraulics, Journal ISAE (India), Vol.XII, No.3-4, pp.1-4

5. McLACHLAN, N.W., (1955). Bessel Functions for Engineers. Second Edition, Oxford University Press, pp.204.

6. CHURCHILL, R.V., Operational Mathematics McGraw Hill Book Company, Second Edition, pp.328.

7 APPENDIX I

Evaluation of Integral I

Resolving into partial fractions

$$I = -\frac{1}{\pi}\int_0^\infty e^{-\mu^2 t} \sin \mu d\left(\frac{1}{\mu+\beta} + \frac{1}{\mu-\beta}\right) d\mu \qquad (1)$$

Setting $\sin \mu d = \sin(\mu+\beta-\beta)d$ and simplifying, we have

$$I = -\frac{1}{\pi}\left[\cos\beta d\left(\int_0^\infty e^{-\mu^2 t} \sin \frac{(\mu\pm\beta)d}{(\mu\pm\beta)} d\mu\right) \mp \sin\beta d\left(e^{-\mu^2 t} \frac{\cos(\mu\pm\beta)d}{(\mu\pm\beta)} d\mu\right)\right] \qquad (2)$$

Setting

$$\sin\frac{(\mu\pm\beta)\ d}{(\mu\pm\beta)} = d\left[\int_0^1 \cos.d(\mu\pm\beta)\alpha\ \ d\alpha\right] \qquad (3)$$

$$\cos\frac{(\mu\pm\beta)}{(\mu\pm\beta)}d = d\left[1 - \int_0^1 \sin d(\mu\pm\beta)\ \alpha\ d\alpha\right] \qquad (4)$$

and substituting in Eq.2 and simplifying and changing the order of integration we have

$$I = -\frac{2d}{\pi}\left[\cos\beta d \int_0^1 \cos d\beta\alpha\left(\int_0^\infty e^{-\mu^2 t} \cos d\mu\alpha\ d\mu\right)d\alpha + \sin\beta d\int_0^1 \sin d\beta\alpha\left(\int_0^\infty e^{-\mu^2 t}\cos d\mu\alpha\ d\mu\right)d\alpha\right] \qquad (5)$$

$$\text{since} \int_0^\infty e^{-\mu^2 t}\cos d\mu\alpha\ d\mu = \tfrac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-d^2\alpha^2/4t} \qquad (6)$$

substituting (6) in (5) and setting $\frac{d^2\alpha^2}{4t} = \lambda_1^2$

and $2\beta t^{\frac{1}{2}} = K_1$ we have

$$I = -\frac{2}{\sqrt{\pi}}\left[\cos\beta d\int_0^{d/2\sqrt{t}} e^{-\alpha_1^2} \cos K_1\alpha_1 d\alpha_1 + \sin\beta d\int_0^{d/2\sqrt{t}} e^{-\lambda_1^2} \sin K_1\alpha_1 d\alpha_1\right] \qquad (7)$$

Expanding $\cos K_1\alpha_1$ and $\sin K_1\alpha_1$ in series form and considering first term of series, we get

$$I = -\cos\beta d\ \mathrm{erf}\left(\frac{d}{2\sqrt{t}}\right) + \frac{K_1 \sin\beta d}{\sqrt{\pi}}\left(e^{-d^2/4t}-1\right) \qquad (8)$$