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The Streamline Direction at a Shock Wave in the Deflection-Speed Plane for a Relaxing Gas

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SUMMARY The effect of relaxation after a curved two-dimensional shock wave in an inviscid, adiabatic gas on the streamline in the δ-V plane is considered. The slope in the δ-V plane of the streamline at the shock is shown to be independent of the relaxation rate and is everywhere non-positive for a straight shock, so that a Crocco point does not exist. If the shock is curved, however the streamline is strongly curved in the relaxation region and eventually leaves it at a direction qualitatively similar to that corresponding to frozen flow. The effect of a constraint, such as the intersection of a body surface with the shock, on the δ-V map is discussed.

1 INTRODUCTION

In a large range of conditions of practical interest shock waves may be treated as mathematical discontinuities, outside which the medium behaves like a perfect gas with translational and (for a diatomic gas) rotational degrees of freedom only. At sufficiently high speeds, however, the temperature after the shock is so high that vibrational degrees of freedom become excited or indeed that the gas dissociates. If the rate of dissociation is fast enough for the relaxation length to be small compared with the length scale of observation, the gas may be considered to be in thermodynamic (including chemical) equilibrium everywhere except in a thin shock. If the relaxation length is comparable with the observation length scale, however, the shock structure consists of a virtually discontinuous translational-rotational shock followed by a nonequilibrium region of finite extent in which vibrational and chemical equilibrium is gradually approached. The mechanism of this approach to equilibrium requires each particle to undergo a large number of collisions with other particles, whereas translational and rotational equilibrium is reached in 2 or 3 collisions.

In the following, the gradient of any flow variable along a streamline just after the translational shock is of particular interest. In the case of frozen flow (no vibrational or chemical change) it will be shown below that this gradient is proportional to the shock curvature, so that it is zero for a straight shock. However, the presence of a finite-rate chemical reaction - which may be thought of as a distributed energy sink in the case of dissociation - causes the gradient to be finite even for a straight shock. The present work aims at studying the implications of this gradient on the features of two-dimensional, inviscid, adiabatic flows through shock waves with finite rates of relaxation. For such flows it is often convenient to map the physical plane into the deflection-speed plane. For this purpose the streamline direction at a shock in the deflection-speed plane is an important variable and its dependence on the relaxation rate is therefore of interest in the present context.

2 DERIVATIVES ALONG THE STREAMLINE

Take x' and y' as coordinates parallel and normal to a shock of curvature $k'(x')$ and making an angle $\phi'(x')$ with the direction of the uniform free stream

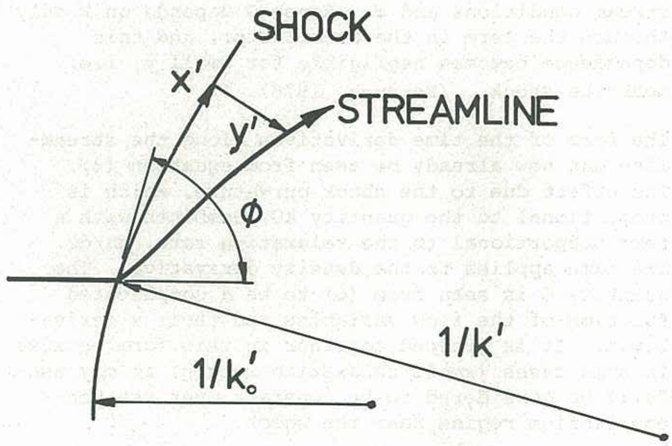

Figure 1 Notation

of velocity V'_∞ (see figure 1). k' is taken to be positive if the shock is convex towards the upstream direction. Let u' and v' be the components of velocity in the x' and y' directions respectively, p' and ρ' be the pressure and density, and h' be the specific enthalpy of the gas. Scale the variables according to

$$\left.\begin{array}{l} h = h'/(V'_\infty)^2,\ p = p'/\rho_\infty (V'_\infty)^2,\ v = v'/V'_\infty, \\ u = u'/V'_\infty,\ \rho = \rho'/\rho'_\infty\ ,\ x = x'k'_o\ ,\ y = y'k'_o\ , \\ \qquad k = k'/k'_o\ , \end{array}\right\} \quad (1)$$

where the subscript ∞ refers to conditions in the free stream and k'_o is the shock curvature at a reference point. Introducing a nonequilibrium variable α (e.g. the dissociation fraction), the caloric equation of state is of the form

$$h = h(p,\rho,\alpha). \quad (2)$$

The conservation equations for momentum, energy and mass in these curvilinear coordinates are

$$\left.\begin{array}{r} uu_x + (1-ky)vu_y - kuv + p_x/\rho = 0, \\ uv_x + (1-ky)\ vv_y + ku^2 + (1-ky)\ p_y/\rho = 0, \\ h + (u^2+v^2)/2 = 0, \\ (\rho u)_x - k\rho v + (1-ky)(\rho v)_y = 0. \end{array}\right\} \quad (3)$$

The subscripts denote partial differentiation. If the third of equations (3) is differentiated with respect to y (this involves terms like $h_p p_y$, $h_\alpha \alpha_y$

etc.) and with respect to x, the set (3) may be solved explicitly for p_y, u_y, v_y, ρ_y in terms of the variables themselves and their x-derivatives and of derivatives of h (see Hornung, 1976). The substantial derivative may be formed by combining x and y derivatives according to

$$\frac{d}{dt} = v\frac{\partial}{\partial y} + \frac{u}{1-ky}\frac{\partial}{\partial x},$$

where t denotes dimensionless time ($t = t'V'_\infty k'_o$ with origin at the shock). The results for the derivatives of p and ρ are

$$\left.\begin{aligned} F\frac{dp}{dt} &= \rho\, h_\alpha \frac{d\alpha}{dt} + k\,\frac{\rho^2 h_\rho}{v}\, G, \\ F\frac{d\rho}{dt} &= \frac{\rho}{v^2}\, h_\alpha \frac{d\alpha}{dt} - k\,\frac{\rho^2}{v}\,(h_p - \frac{1}{\rho})\, G, \end{aligned}\right\} \quad (4)$$

where $F = 1 - \rho(\frac{h_\rho}{v^2} + h_p)$, (5)

and $G = \{v^2+u^2+ \frac{uv_x}{k} - \frac{u}{\rho v}\frac{p_x}{k} - v\frac{u_x}{k}\}/(1-ky)$. (6)

Note that the x-derivatives occurring in G are all proportional to k, so that terms like p_x/k are independent of k, and are functions only of free stream conditions and ϕ. Hence G depends on k only through the term in the denominator, and this dependence becomes negligible for small y, i.e. near the shock. (Hornung, 1976).

The form of the time derivatives along the streamline can now already be seen from equation (4). The effect due to the shock curvature, which is proportional to the quantity kG, combines with a term proportional to the relaxation rate, dα/dt. The same applies to the density derivative. The quantity G is seen from (6) to be a complicated function of the flow variables and their x derivatives. It is grouped together in this form because in some cases (small relaxation length) it may usefully be considered to be constant over the non-equilibrium region near the shock.

To illustrate the two limits k = 0 and dα/dt = 0, it is instructive to form the quotient of the two equations (4):

$$\frac{dp}{d\rho} = v^2\,\frac{h_\alpha d\alpha/dt + k\,\rho\, h_\rho\, G/v}{h_\alpha d\alpha/dt - k\,\rho\,(h_p - 1/\rho)G/v}. \quad (7)$$

For a straight shock in relaxing flow (k = 0, dα/dt ≠ 0) dp/dρ along the streamline is seen to be independent of the relaxation rate dα/dt and is equal to v^2. Conversely, for a curved shock in frozen flow (k≠0, dα/dt = 0) dp/dρ is independent of k and G, and is given by

$$\frac{dp}{d\rho} = -\frac{h_\rho}{h_p - 1/\rho} = a_f^2, \quad (8)$$

where a_f is the frozen speed of sound. This expresses the expected isentropic flow condition.

In a similar way substantial derivatives of other quantities may be formed. In particular, the condition for the conservation of momentum in the streamline direction requires

$$V\frac{dV}{dt} = -\frac{1}{\rho}\frac{dp}{dt}, \quad (9)$$

where $V^2 = v^2 + u^2$. The streamline curvature, dδ/ds, may be determined by differentiating the streamline deflection

$$\delta = \phi - \text{artan}\,\frac{v}{u}, \quad (10)$$

with respect to the distance along the streamline, s(ds=Vdt) to give after some manipulation

$$F\frac{d\delta}{ds} = \frac{u}{vV}\left\{\frac{h_\alpha}{V^2}\frac{d\alpha}{dt} + \frac{k\rho h_\rho}{v}\left[\frac{G}{V^2} - \frac{v}{u}\frac{p_x}{\rho k}\left(\frac{1}{a_f^2} - \frac{1}{v^2}\right)\right]\right\}. \quad (11)$$

3 MAPPING FROM THE PHYSICAL TO THE δ-V PLANE

It is convenient to introduce a relaxation rate variable

$$\omega = -\frac{v}{k\rho h_\rho V^2}\, h_\alpha \frac{d\alpha}{dt}. \quad (12)$$

From (9) and (11) the quantity dδ/dV may be formed:

$$\frac{d\delta}{dV} = -\frac{u}{Vv}\left[\omega - G + \frac{vV^2}{u}\cdot\frac{p_x}{\rho k}\left(\frac{1}{a_f^2} - \frac{1}{v^2}\right)\right]/(\omega - G). \quad (13)$$

This gives the slope in the δ-V plane of a streamline near the shock wave. The streamline may now be mapped into the δ-V plane, and relaxing and frozen flows may be compared for straight and curved shocks.

Consider first a frozen flow after a straight shock in a uniform free stream. The conditions after the shock are uniform, δ and V are both constant, and the whole of the flow downstream of a shock of given incidence ϕ, maps into a single point (P, say, in figure 2). The free stream maps into the point F at (1,0). As ϕ is changed, P traces out the locus of conditions after the shock as indicated in figure 2. V is seen to be double-valued for a given δ, the maximum deflection point M separating the strong shock branch from the weak one. At some point S to the right of M the sonic speed a_f is equal to V. At the point N, $\phi = \frac{\pi}{2}$, i.e. the shock is normal to the free stream direction.

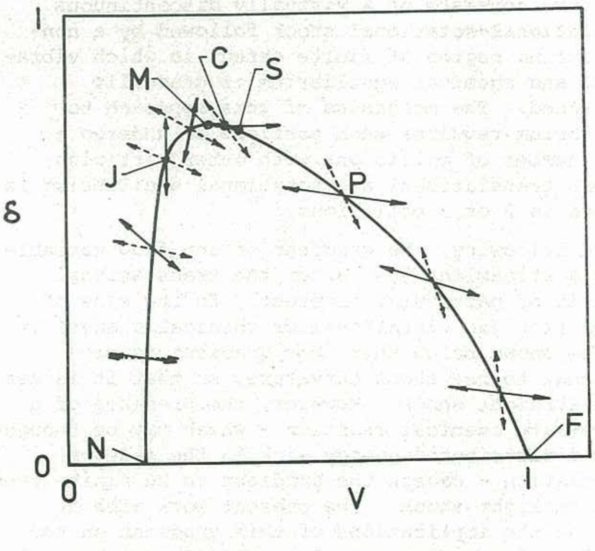

Figure 2 Locus of shock conditions in the δ-V plane. P - general point on locus, F - free stream, N - normal shock, S - sonic point, C - Crocco point, M - maximum deflection, J - point where G = 0. Arrows indicate direction of streamline at a curved shock in frozen flow. Dotted arrows indicate directions of streamlines at a straight shock in relaxing flow.

Now let the shock be curved and the flow remain frozen. The flow will then accelerate along the streamline, so that the whole of the streamline after the shock no longer maps into a single point, but into a line. Equation (13) is able to give the slope of this line at the shock as

$$\left[\frac{d\delta}{dV}\right]_{\omega=0} = -\frac{u}{Vv}\left[1 - \frac{vV^2}{uG}\cdot\frac{p_x}{\rho k}\left(\frac{1}{a_f^2} - \frac{1}{v^2}\right)\right], \quad (14)$$

where ω has been put = 0 for frozen flow. The expression on the right of (14) has three zeros and one pole in the range of the curve shown in figure 2. Two of the zero's occur when u = 0 at N and F and are of no particular interest. The third corresponds to the zero of the expression in square brackets and occurs at the familiar Crocco point C which always lies between S and M (see e.g. Guderley 1962). The pole occurs at J, where G passes through zero. Between F and C, $(d\delta/dV)_{\omega=0} < 0$ but finite. Between C and J, $(d\delta/dV)_{\omega=0} > 0$, and between J and N it is negative again. This behaviour is indicated in figure 2 by the thin arrows representing the streamline direction at the shock in frozen flow.

The above results are, of course, well established and discussed in detail in textbooks on gasdynamics. However, (13) may now be used to see how the streamline direction is modified by the presence of a relaxation process. Consider first the straight shock with finite relaxation rate, that is, the limit $\omega = \infty$. Clearly, (13) gives

$$\left[\frac{d\delta}{dV}\right]_{\omega=\infty} = -\frac{u}{Vv} . \quad (15)$$

Although this result could have been obtained directly by differentiating (10) with respect to V, it is of some importance, because it shows that in the limit $\omega = \infty$ the streamline slope in the δ-V plane is independent of the relaxation rate, and, since u, V and v are all $\geqslant 0$, $(d\delta/dV)_{\omega=\infty} \leqslant 0$ everywhere. The streamline directions for $\omega = \infty$ are indicated in figure 2 as dotted arrows. Moreover, since k = 0 for a straight shock, the term (1-ky) which has been considered equal to 1 in the derivation of (13) is exactly 1 throughout the flow, and (15) applies throughout the relaxation layer. That part of the streamline which is in the relaxation layer is mapped into a line in the δ-V plane, one end point of which corresponds to the point on the shock and the other representing the whole of the (straight) streamline downstream of the relaxation layer, that is, the equilibrium condition after a straight shock. Let this point be labelled E, see figure 3. If the relaxation is endothermic, such as for example in dissociating flow, E lies outside the curve P, and conversely, for exothermic relaxation, such as occurs in a detonation wave, E lies inside P. By changing the shock incidence, E is made to trace out a curve corresponding to the locus of equilibrium conditions after a straight shock. This curve is shown for the case of endothermic relaxation in figure 3.

4 RELAXING THE CONDITION $\omega = \infty$

Equation (13) may be written for $\omega \to \infty$ as

$$\left[\frac{d\delta}{dV}\right]_{\omega\to\infty} = -\frac{u}{Vv}\left[1 + \frac{1}{\omega}\cdot\frac{vV^2}{u}\frac{p_x}{\rho k}\left(\frac{1}{a_f^2} - \frac{1}{v^2}\right)\right] + O\left(\frac{1}{\omega^2}\right). \quad (16)$$

Note that to first order in $1/\omega$ this is independent of G. This limit corresponds to the case when the relaxation distance is small compared to the shock radius of curvature. However, as the distance from the shock increases, the relaxation rate and therefore ω decreases also, so that (16) is no longer appropriate but must be replaced by the limit $\omega \to 0$,

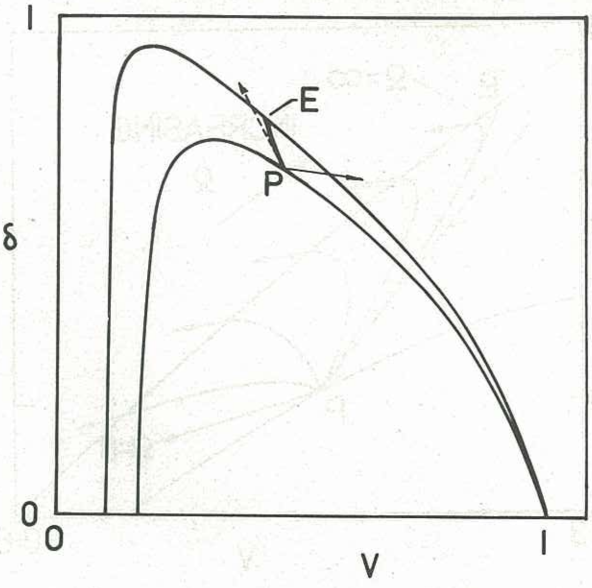

Figure 3 Locus of equilibrium conditions (E) after a straight shock in relaxing flow. Full and dotted arrows indicate streamline directions at the shock (P) for $\omega = 0$ and $\omega = \infty$ respectively.

$$\left[\frac{d\delta}{dV}\right]_{\omega\to 0} = -\frac{u}{vV}\left[1 - \frac{vV^2}{uG}\cdot\frac{p_x}{\rho k}\left(\frac{1}{a_f^2} - \frac{1}{v^2}\right)(1+\omega)\right] + O(\omega^2), \quad (17)$$

which asymptotes to (14). For a slightly curved shock with a fast relaxation rate the streamline therefore leaves the shock in a direction close to the dotted arrow corresponding to $\omega = \infty$ and then curves over to the direction represented by $\omega = 0$, on the equilibrium curve (E). Since the shock is curved, the flow along the streamline continues to accelerate in the equilibrium region and is therefore not mapped into a point but into a line. This line must leave the equilibrium curve, E, at a slope given by (14), so that the points M, C, S and J have equivalents on the equilibrium curve.

Let the value of ω on the shock be Ω. The behaviour of the streamline in the relaxation region may then be described qualitatively as shown in figure 4 as a function of Ω. The limit $\Omega = \infty$ (straight shock) gives the curve described by (15) and the limit $\Omega = 0$ (frozen flow) is described by (14). Between these limits figure 4 shows a number of intermediate curves which must asymptote to (14) at the equilibrium curve. These intermediate curves may be thought of as flows after shocks in a given gas with various values of curvature.

It remains to point out that figures 4 and 5 are drawn for endothermic relaxation with a convex shock or for exothermic relaxation with a concave shock, i.e. for positive ω. The above equations apply equally for negative ω, when the equilibrium curve lies inside the shock curve and the other arrow direction applies throughout.

5 CONSTRAINED SHOCK POINTS

So far, only free shock points have been discussed, that is, points on a shock wave which do not coincide with the intersection of a body or another shock wave with the shock under consideration. An interesting example of the case of a constrained shock point is the case of an attached shock at the leading edge of a sharp body. At the leading edge,

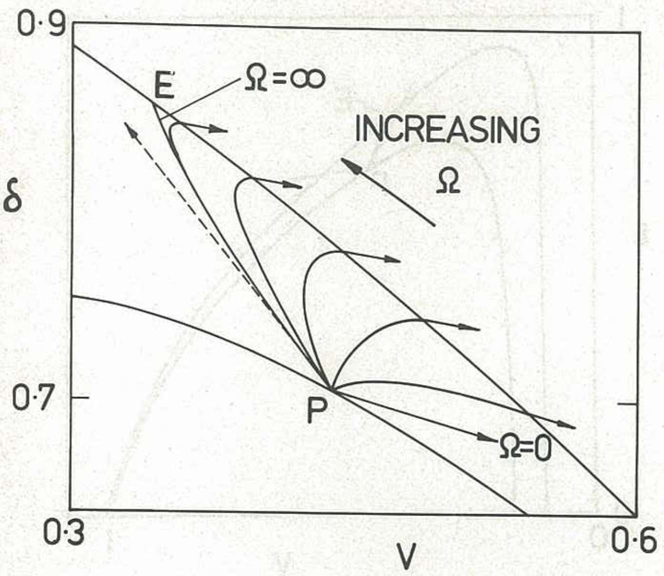

Figure 4 Behaviour of the streamline in the relaxation region for various values of Ω. $\Omega = \infty$: straight shock, $\Omega = 0$: frozen flow. Note that streamlines leave the equilibrium curve (E) at a direction like that for $\Omega = 0$. The numbers on the δ and V axes are shown merely to indicate the approximate region. Significance of dotted arrow and P as in figure 2.

the streamline curvature is constrained to be equal to the body curvature. The streamline curvature must also satisfy equation (11), however. Evidently both conditions cannot be satisfied by a straight shock unless the body curvature is just right. The flow satisfies the two conditions by adjusting ω to suit. That is, for a given relaxation rate, the shock curvature adjusts itself to provide the right streamline curvature to match the body. For example, for a straight wedge, $d\delta/ds = 0$, and (11) gives

$$\omega = \frac{G}{V^2} - \frac{v}{u}\frac{p_x}{\rho k}\left(\frac{1}{a_f^2} - \frac{1}{V^2}\right). \quad (18)$$

This result can be used to measure relaxation rates by measuring the shock curvature at the leading edge of a wedge, as has been pointed out by Becker (1972). It has been used to measure the dissociation rates of nitrogen up to 15000 K by Kewley and Hornung (1974). To illustrate this behaviour, the mapping of the flow near the leading edge of a wedge is shown in figure 5.

Other examples of constrained shock points include the intersection of two shock waves, and three-shock points such as occur in Mach reflexion of shocks. The influence of relaxation on such points is more complicated than the wedge tip and is beyond the scope of the present considerations. The means by which the flow adjusts itself to the constraint, however, must be by adjusting ω.

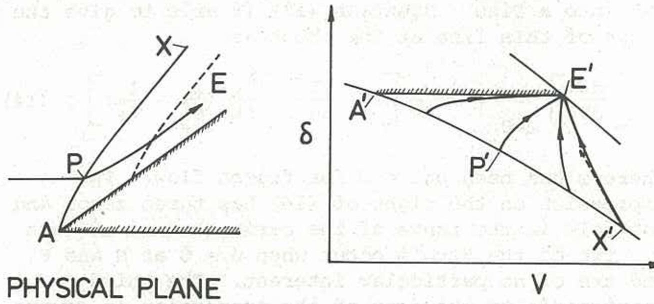

Figure 5 Flow near the constrained shock point at the leading edge of a wedge with an attached shock. Conditions in the equilibrium region E are uniform, so that it is mapped into a single point E' in the δ-V plane. At the constrained point A, ω adjusts itself to give zero slope for the line A'E'. At X the shock has approached constant ϕ, so that the streamline leaves X' in the direction of the dotted arrow. The whole of the flow above X is mapped into the line X'E'.

6 CONCLUSIONS

Two important results of considering relaxing flows in the δ-V plane are, firstly, that the slope of the streamline $d\delta/dV$ after a straight shock is independent of the relaxation rate and is non-positive for all values of the shock incidence and, secondly, that constraints imposed by a second shock or a body intersecting the shock are satisfied by the value of ω adjusting itself to match them. The flow achieves this by adjusting the shock curvature. Thinking of relaxing flows in the δ-V plane greatly aids the understanding of complex relaxing flow situations such as shock reflexion or shock detachment, both of which differ significantly from the corresponding frozen flow situations. These situations are discussed in some detail elsewhere (see Hornung and Kychakoff 1977, Hornung and Smith, 1978).

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