6th Australasian Hydraulics and
Fluid Mechanics Conference
Adelaide, Australia, 5-9 December 1977

A Case of Steady Plane Flow of an Ideal Gas with Infinite Electrical Conductivity in a Magnetic Field

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SUMMARY An analytical solution to the magnetohydrodynamic equations of steady plane flow of an ideal gas with infinite electrical conductivity is found and discussed, assuming that (i) the physical quantities appearing in the equations are functions of the distance from the origin (of the physical space) alone, and (ii) the flow velocity is parallel to the magnetic field. The discussion, preceded by some additional derivations, is in particular concerned with possible types of the flow and its subdivisions into different regions and/or regimes. Most of the conclusions reached are related to analytical results involving three characteristic velocities, as well as to a certain parameter which is a product of some non-negative constant and the stagnation density. For all types of the flow, there is a vacuum around the origin.

NOTATION

For any vector, $\underset{\sim}{a}$ say, $a \equiv |\underset{\sim}{a}|$.

c_1, c_2, c_3, c_5 - arbitrary constants, $c_2 \neq 0$, $c_3 > 0$; $c_4 \equiv c_1/c_2$;

M - local Mach number;

r - distance from the origin of the physical space;

$\tilde{r} \equiv r v_{max} \rho_* / |c_2|$;

v_{max} - maximum velocity of the flow;

$\tilde{v} \equiv v/v_{max}$;

v_a - Alfvén velocity corresponding to H (see also (25,ii));

$v_b \equiv v_a(1 - M^2)^{\frac{1}{2}}$ - a characteristic velocity;

v_s - acoustic velocity (see also (25,i));

$\alpha \equiv (c_5\rho_*/c_2)^2$; $\beta \equiv (c_4^2/4\pi)\rho_*$;

γ - isentropic constant;

ρ_* - stagnation density (see also (14));

$\tilde{\rho} \equiv \rho/\rho_*$; $\tilde{\rho}_a$, $\tilde{\rho}_b$, $\tilde{\rho}_s$ - see (29), (34)-(35), and (27), respectively;

$' \equiv \frac{d}{dr}$;

the subscripts r and θ refer respectively to radial and transverse components of a quantity.

The remaining notation is either obvious or of lesser importance.

1 THE PROBLEM AND THE GOVERNING EQUATIONS

The steady motion of an ideal gas with infinite electrical conductivity in the presence of a magnetic field, can be described under certain conditions (see Landau and Lifshitz (1960)) by a system of equations consisting of:

(i) the field equations

$$\text{div}\, \underset{\sim}{H} = 0, \quad \text{curl}\,(\underset{\sim}{H} \times \underset{\sim}{v}) = \underset{\sim}{0}; \tag{1}$$

(ii) the fluid mechanics equations

$$\text{div}(\rho \underset{\sim}{v}) = 0, \tag{2}$$

$$(\underset{\sim}{v}.\text{grad})\underset{\sim}{v} = -\frac{\text{grad}\, p}{\rho} + \frac{(\text{curl}\, \underset{\sim}{H}) \times \underset{\sim}{H}}{4\pi\rho}; \tag{3}$$

and (iii) the conservation of entropy equation ≡ the adiabatic condition.

In this paper, we restrict our attention to the case when all physical quantities involved are functions of a single variable r which represents the distance from the origin of the physical space. Furthermore, it is convenient to assume in the foregoing considerations that the velocity and magnetic field vectors are parallel. (That this assumption does not reduce the generality follows from the use of a particular coordinate system (Landau and Lifshitz, 1960 p 230) and some conclusions for plane flows of a gas with infinite electrical conductivity, due to Kogan (1959), §1.)

Taking into account these assumptions, and introducing the plane polar coordinates (r,θ), we reduce the system (1)-(3) plus the adiabatic condition to the following form:

$$rH_r = c_1, \quad H_r v_\theta - H_\theta v_r = 0; \tag{4}$$

$$\rho r v_r = c_2, \tag{5}$$

$$\frac{v_\theta^2}{r} - v_r v_r' = \frac{p'}{\rho} + \frac{H_\theta/r}{4\pi\rho}(rH_\theta)', \tag{6}$$

$$\frac{v_r v_\theta}{r} + v_r v_\theta' = \frac{H_r/r}{4\pi\rho}(rH_\theta)'; \tag{7}$$

$$p/\rho^\gamma = c_3; \tag{8}$$

where ' stands for $\frac{d}{dr}$; c_1, c_2, c_3 are constants, $c_2 \neq 0$, $c_3 > 0$. (For the validity of (8), see Kogan (1959).)

2 BASIC ANALYTICAL RESULTS

2.1 General results

From (4) and (5), we have

$$H_r = c_4 \rho v_r, \quad H_\theta = c_4 \rho v_\theta, \quad c_4 \equiv \frac{c_1}{c_2}. \tag{9}$$

Therefore

$$H = |c_4| \rho v, \tag{10}$$

and subsequently the ratio of magnetic and kinetic energy densities, i.e. $(H^2/8\pi)/(\rho v^2/2)$, satisfies the relation

$$(H^2/8\pi)/(\rho v^2/2) = (c_4^2/4\pi)\rho. \tag{11}$$

Using (9), we can replace (7) by

$$\left[r v_\theta \left(1 - \frac{c_4^2}{4\pi}\rho\right)\right]' = 0,$$

thus implying

$$r v_\theta \left(1 - \frac{c_4^2}{4\pi}\rho\right) = c_5, \quad c_5 \equiv \text{const.} \tag{12}$$

It has been shown by Kogan (1959) that the Bernoulli equation of classical gasdynamics remains valid for the gas flow under consideration, and so, introducing the maximum velocity of the flow, v_{max}, (Kogan, 1959) and using (8), one may write

$$v^2 = v_{max}^2 - \frac{2\gamma c_3}{\gamma - 1} \rho^{\gamma-1}. \tag{13}$$

Now, let ρ_* be the stagnation density, which in view of (13) is given by

$$\rho_* = \left[\frac{2\gamma c_3}{(\gamma - 1) v_{max}^2}\right]^{\frac{1}{1-\gamma}}, \tag{14}$$

and let $\tilde{\rho}$ and $\tilde{v}$ denote the dimensionless quantities defined by

$$\text{i) } \tilde{\rho} \equiv \rho/\rho_* \quad \text{and} \quad \text{ii) } \tilde{v} \equiv v/v_{max}. \tag{15}$$

Then we may rewrite (13) as

$$\tilde{v}^2 = 1 - \tilde{\rho}^{\gamma-1}. \tag{16}$$

Remark. Since $\tilde{v}$ is within the interval $[0,1]$, so is $\tilde{\rho}$ - by (16).

Obviously

$$\tilde{v}^2 = \tilde{v}_r^2 + \tilde{v}_\theta^2 \tag{17}$$

where

$$\text{i) } \tilde{v}_r \equiv v_r/v_{max} \quad \text{and} \quad \text{ii) } \tilde{v}_\theta \equiv v_\theta/v_{max}. \tag{18}$$

Also it is easy to see from (5), (18,i) and (15,i) that

$$\tilde{v}_r^2 = (\tilde{\rho}\tilde{r})^{-2} \tag{19}$$

where

$$\tilde{r} \equiv r v_{max} \rho_* / |c_2|, \tag{20}$$

while from (12), (18,ii), (15,i) and (20),

$$\tilde{v}_\theta^2 = \alpha[(1 - \beta\tilde{\rho})\tilde{r}]^{-2} \tag{21}$$

where

$$\text{i) } \alpha \equiv \left(\frac{c_5 \rho_*}{c_2}\right)^2 \quad \text{and} \quad \text{ii) } \beta \equiv \frac{c_4^2}{4\pi}\rho_*. \tag{22}$$

Substitution of $\tilde{v}_r^2$ and $\tilde{v}_\theta^2$, as given by (19) and (21), into (17) yields

$$\tilde{v}^2 = \frac{\alpha\tilde{\rho}^2 + (1 - \beta\tilde{\rho})^2}{\tilde{r}^2\tilde{\rho}^2(1 - \beta\tilde{\rho})^2}, \tag{23}$$

and moreover, on the basis of (16) and (23), we obtain

$$\tilde{r}^2 = \frac{\alpha\tilde{\rho}^2 + (1 - \beta\tilde{\rho})^2}{\tilde{\rho}^2(1 - \beta\tilde{\rho})^2(1 - \tilde{\rho}^{\gamma-1})}. \tag{24}$$

Note that the mapping $\tilde{r} \mapsto \tilde{\rho}$ is not a one-to-one correspondence, i.e. the function $\tilde{\rho} = \tilde{\rho}(\tilde{r})$ is multivalued.

2.2 Results involving characteristic velocities

It will be useful for the following to find the values of $\tilde{\rho}$ for which (i) $v = v_s \equiv$ the acoustic velocity, or (ii) $v = v_a \equiv$ the Alfvén velocity corresponding to H.

By definition,

$$\text{i) } v_s \equiv (dp/d\rho)^{\frac{1}{2}} \quad \text{and} \quad \text{ii) } v_a \equiv (4\pi\rho)^{-\frac{1}{2}} H. \tag{25}$$

Using (8), (15,i) and (14), we have from (25,i)

$$v_s^2 = \frac{\gamma - 1}{2} v_{max}^2 \tilde{\rho}^{\gamma-1}, \tag{26}$$

and substitution of $\tilde{v}^2 = \tilde{v}_s^2 \equiv v_s^2/v_{max}^2$ into (16), plus some evident observation, imply

$$v = v_s \quad \text{iff} \quad \tilde{\rho} = \tilde{\rho}_s \equiv \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}}. \tag{27}$$

Using (10), (15,i) and (22,ii), we obtain from (25,ii)

$$v_a^2 = \beta\tilde{\rho}v^2, \tag{28}$$

and therefore

$$v = v_a \quad \text{iff} \quad \tilde{\rho} = \tilde{\rho}_a \equiv \beta^{-1}. \tag{29}$$

Remark. In view of Remark of Section 2.1, (29) is feasible only if $\beta \geq 1$.

As Kogan (1959) has pointed out, in addition to v_a and v_s there is the third important characteristic velocity

$$v_b \equiv v_a\left(1 - M^2\right)^{\frac{1}{2}}, \tag{30}$$

where $M \equiv v/v_s$ is the local Mach number. On the basis of (16) and (15,ii)

$$M = \left(\frac{1 - \tilde{\rho}^{\gamma-1}}{1 - \tilde{\rho}_s^{\gamma-1}}\right)^{\frac{1}{2}}. \tag{31}$$

Thus, using (28) and (31), we may write instead of (30),

$$v_b^2 = \beta\tilde{\rho}\,\frac{\tilde{\rho}^{\gamma-1} - \tilde{\rho}_s^{\gamma-1}}{1 - \tilde{\rho}_s^{\gamma-1}}\,v^2. \tag{32}$$

Clearly from (32),

$$v_b \text{ is real} \quad \text{iff} \quad \tilde{\rho} \geq \tilde{\rho}_s. \tag{33}$$

Also from (32),

$$v = v_b \quad \text{iff} \quad \tilde{\rho} = \tilde{\rho}_b, \tag{34}$$

where $\tilde{\rho}_b$ is determined by

$$\beta\tilde{\rho}_b\,\frac{\tilde{\rho}_b^{\gamma-1} - \tilde{\rho}_s^{\gamma-1}}{1 - \tilde{\rho}_s^{\gamma-1}} = 1. \tag{35}$$

In view of (33) and Remark of Section 2.1, $\tilde{\rho}_b$ is within the interval $[\tilde{\rho}_s,1]$. Therefore, since the right-hand side of (35), considered as a function of $\tilde{\rho}_b$, is strictly increasing, we conclude from (34) and (35) that

$$v \text{ can be equal to } v_b \quad \text{iff} \quad \beta \in [1,\infty), \tag{36}$$

and that

the mapping $\beta \mapsto \tilde{\rho}_b$, given by (35) and such that $[1,\infty) \to (\tilde{\rho}_s,1]$, is bijective. (37)

3 DISCUSSION AND CONCLUSIONS

As is easily seen from (24), $\tilde{r} \to \infty$ if and only if (i) $\tilde{\rho} \to 0$ or (ii) $\tilde{\rho} \to 1$ or (iii) $\tilde{\rho} \to \beta^{-1}$, that is, in view of (16), if and only if (i) $\tilde{v} \to 1$ or (ii) $\tilde{v} \to 0$ or (iii) $\tilde{v} \to 1 - \beta^{1-\gamma}$. By Remark of Section 2.1, $\tilde{\rho}$ is within $[0,1]$ and therefore, taking into account that $\alpha \geq 0$ by (22,i), we also conclude from (24) that $\inf_{\alpha,\beta,\tilde{\rho}} \tilde{r} \equiv \tilde{r}_{min} > 0$. (It may be mentioned here that, in view of (12), $\alpha = 0$ corresponds to the cases when the flow velocity is purely radial or when the magnetic and kinetic energy densities are equal; see (11).) Thus for any choice of (permissible) α and β there is a vacuum in the interior of a disc of radius $r = |c_2|\tilde{r}_{min}/(\rho_* v_{max})$ about the origin. The conclusion in particular removes a need to consider the flow in a neighbourhood of the "formal" singularity at $r = 0$.

These general observations and conclusions can be supplemented by the following ones:

(a) $v_a < v_s$ or $v_a > v_s$ depending on whether $\beta^{-1} > \tilde{\rho}_s$ or $\beta^{-1} < \tilde{\rho}_s$ (see (27), (29), and (16); for completeness also note Remark of Section 2.2);

(b) v_b, assumed real, is never greater than v_s (see (16) and (33));

(c) sup H is attained when $v = v_s$ (to show this, find the stationary points of $\rho^2 v^2$ where v^2 is given by (13), and take into account (10) and (27)); this observation is not restricted only to the case considered in the present paper.

The abovementioned types (i) and (ii) of the flow are possible for the whole range of β, while the type (iii) is only possible if $\beta > 1$ and in such a case (i.e., for $\beta > 1$), in accordance with (29), $\tilde{\rho} = \beta^{-1}$ when $v = v_a$. For (i) and (ii), in view of (10), $\lim_{r\to\infty} H = 0$. Moreover, for (i), the ratio of the magnetic and kinetic energy densities also tends to zero if $r \to \infty$ (see (11)).

If $\beta < 1$, then by (36) and Remark of Section 2.2 there are no points in the flow where $v = v_a$ or $v = v_b$. (If $\beta \geq 1$: see (34) and (37) for the case $v = v_b$; of course, in view of (29), a statement similar to (37) is true in the case $v = v_a$.) As in classical gasdynamics, the subsonic region (i.e. where $\tilde{\rho} > \tilde{\rho}_s$, by (31)) is elliptic, while the supersonic region (i.e. where $\tilde{\rho} < \tilde{\rho}_s$, by (31)) is hyperbolic. (We use the terminology as in Kogan (1959); note the omission (by the translator) of the word "elliptic" on p 94, 13th line from the bottom.)

Further discussion and conclusions, based in particular on numerical computations and graphical illustrations of the analytical results obtained in this paper, will be presented elsewhere.

4 REFERENCES

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