

The Application of a Finite Element Method to Compressible, Inviscid Flow

C. A. J. FLETCHER

Department of Mechanical Engineering, University of Sydney

SUMMARY Alternative formulations for external, inviscid, compressible flows are compared. A novel Galerkin formulation based on the conservation form of the equations is described. The use of reduced integration to evaluate certain key integrals in a related model problem produces a tenfold improvement in computational efficiency. The source of this improvement indicates a new finite element formulation. The use of Newton's method to solve the non-linear, governing algebraic equations associated with the Galerkin formulation is found to be unsatisfactory. A least-squares formulation based on the conservation form of the governing equations does not require the use of Newton's method and produces satisfactory results for the flow about typical aerofoils. Comparisons are made with experimental and other computational results.

1 VELOCITY POTENTIAL vs PRIMITIVE VARIABLES

In the present paper finite element formulations suitable for external, inviscid, compressible flows will be considered. The present work is a stepping-stone towards the treatment of inviscid transonic flow. Most previous attempts at solving subcritical flow using a finite element formulation, e.g. Labrujere (1974); Periaux (1975), have used a velocity potential, ϕ , and the local speed of sound, a , as dependent variables. The governing equations in two dimensions are then

$$(a^2 - \phi_x^2) \cdot \phi_{xx} - 2 \cdot \phi_x \cdot \phi_y \cdot \phi_{xy} + (a^2 - \phi_y^2) \cdot \phi_{yy} = 0 \quad (1)$$

and

$$a^2 + 0.5 (\gamma - 1) \cdot (\phi_x^2 + \phi_y^2) = a_o^2 \quad (2)$$

γ is the specific heat ratio and a_o is the stagnation value of the sound speed. At first sight this approach appears attractive because only one unknown is required at each node in a finite element grid. (2) can be treated as a local algebraic relation which adjusts the value of a at each step of the iteration. Also the form of (1) lends itself directly to iterative solution of the successive over-relaxation (SOR) type with the advantage of guaranteed convergence from any starting values.

However a solution procedure utilizing (1) and (2) has certain disadvantages. The final solution is in terms of the velocity potential whereas the desired quantity is the pressure at the external surface of the body. Thus the final solution must be differentiated numerically to obtain useful information. The direct SOR iterative procedure becomes more and more inefficient as local sonic conditions are approached. Thus special iterative techniques must be introduced if the formulation is to handle transonic flow. Because (1) is highly nonlinear the application of a finite element formulation produces a large number of cross-terms that must be manipulated at each step of the iterative process. This large overhead makes a substantial contribution to the overall computation time.

The present treatment of the inviscid, compressible flow problem operates in terms of the primitive

variables and expresses the governing equations in conservation form. Thus

$$(\rho u)_x + (\rho v)_y = 0 \quad (3)$$

$$(\rho u^2 + p)_x + (\rho uv)_y = 0 \quad (4)$$

$$(\rho uv)_x + (\rho v^2 + p)_y = 0 \quad (5)$$

$$p = \rho a^\gamma \quad (6)$$

It is believed that this is the first time a primitive variable finite element formulation has been applied to the full equations of motion governing compressible, inviscid flow. Examination of (3), (4) and (5) indicates that they have the same form and are linear in terms of the various groups of variables. This situation may be contrasted with (1). Equations (3) to (5) are also applicable to local areas of supersonic flow. The final solution for the pressure is obtained directly. Possible disadvantages of this formulation are the requirement of three unknowns per node and the need for special iterative techniques depending on the particular finite element formulation considered.

2 A GALERKIN FINITE ELEMENT FORMULATION

As with any finite element formulation analytic representations for the dependent variables are introduced. Here these are

$$\begin{Bmatrix} \rho u \\ \rho v \\ \rho u^2 \\ \rho uv \\ \rho v^2 \\ p \end{Bmatrix} = \sum_j N_j(x, y) \cdot \begin{Bmatrix} (\overline{\rho u})_j \\ (\overline{\rho v})_j \\ (\overline{\rho u^2})_j \\ (\overline{\rho uv})_j \\ (\overline{\rho v^2})_j \\ \overline{p}_j \end{Bmatrix} \quad (7)$$

where N_j is a quadratic, serendipity rectangular shape function appropriate to the j^{th} node and — indicates the nodal values of the different variables. A unique feature of the present formulation is that the analytic representation is

applied to groups of variables, e.g. ρu^2 , rather than single variables, e.g. u . An immediate advantage is that no cross-terms requiring expensive double summations occur; thus computational efficiency is enhanced. An apparent disadvantage, that will be dealt with later, is that there are more unknowns than equations!

Substitution of the representations (7) into the governing equations, (3) to (5), produces the following residuals,

$$R^{(1)} = \sum_j \frac{\partial N_j}{\partial x} \cdot (\overline{\rho u})_j + \sum_j \frac{\partial N_j}{\partial y} \cdot (\overline{\rho v})_j \quad (8)$$

$$R^{(2)} = \sum_j \frac{\partial N_j}{\partial x} \cdot (\overline{\rho u^2} + \bar{p})_j + \sum_j \frac{\partial N_j}{\partial y} \cdot (\overline{\rho uv})_j \quad (9)$$

$$R^{(3)} = \sum_j \frac{\partial N_j}{\partial x} \cdot (\overline{\rho uv})_j + \sum_j \frac{\partial N_j}{\partial y} \cdot (\overline{\rho v^2} + \bar{p})_j \quad (10)$$

Application of the Galerkin method (Fletcher, 1977b) i.e.

$$\iint N_i \cdot R^{(k)} \cdot dx \cdot dy = 0, \quad k = 1, 3; \quad i = 1, n, \quad (11)$$

produces algebraic relationships of the following form

$$\left. \begin{aligned} \sum_j a_{ij} \cdot (\overline{\rho u})_j + \sum_j b_{ij} \cdot (\overline{\rho v})_j &= 0 \\ \sum_j a_{ij} \cdot (\overline{\rho u^2} + \bar{p})_j + \sum_j b_{ij} \cdot (\overline{\rho uv})_j &= 0 \\ \sum_j a_{ij} \cdot (\overline{\rho uv})_j + \sum_j b_{ij} \cdot (\overline{\rho v^2} + \bar{p})_j &= 0 \end{aligned} \right\} \quad i=1, n \quad (12)$$

where

$$a_{ij} = \iint N_i \cdot \frac{\partial N_j}{\partial x} \cdot dx \cdot dy \quad (13)$$

and

$$b_{ij} = \iint N_i \cdot \frac{\partial N_j}{\partial y} \cdot dx \cdot dy \quad (14)$$

Equations (12) are attractive because of their simple form and because the same algebraic coefficients occur throughout. With the introduction of an isoparametric formulation the evaluation of (13) and (14) can be carried out, once and for all, on a dummy element (Fletcher, 1976). This is desirable so that the computation time may be reduced as much as possible.

3 CALCULATION OF a_{ij} , b_{ij}

The integrations in (13) and (14) are carried out numerically. For rectangular elements Gauss quadrature formulae are appropriate. At first sight it would appear desirable that the order of the Gauss quadrature formula should be sufficiently high that the integrations in (13) and (14) are carried out exactly. However there is considerable empirical evidence from both structural applications (Zienkiewicz, 1976) and fluid flow applications (Fletcher, 1977a) that more accurate final results may be obtained if the order of numerical integration is reduced.

The evaluation of a_{ij} and b_{ij} , in (13) and (14), would be exact if a 3x3 Gauss quadrature formula

were used. For a model problem of inviscid, incompressible flow about a two-dimensional cylinder the same expressions for a_{ij} and b_{ij} arise

although the governing equations are different. For this problem the use of a reduced integration (i.e. a 2x2 Gauss quadrature formula) produces a dramatic improvement. Figure 1 shows the tangential velocity at the surface of a circular cylinder obtained with exactly the same grid but with differing orders of numerical integration. RMS differences between the finite element solutions

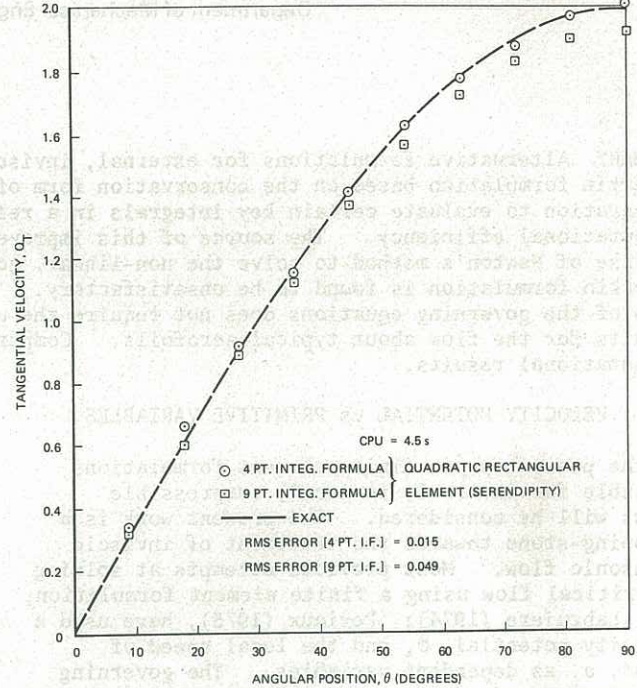


Figure 1 Comparison of reduced integration and exact integration for incompressible flow about a circular cylinder

and the exact solution at the body surface have been computed and indicate the solution using reduced integration is approximately three times as accurate. To achieve the same accuracy using exact numerical integration required a ten-fold increase in computation time.

It can be shown (Fletcher, 1977c) that the use of a 2x2 Gauss quadrature formula applied to the integrals in (13) and (14) is equivalent to replacing the term $\frac{\partial N_j}{\partial x}$ by its least-squares fit over each element, i.e.

$$\iint_{2x2} N_i \cdot \frac{\partial N_j}{\partial x} \cdot dx \cdot dy \equiv \iint N_i \cdot \frac{\partial N_j}{\partial x}_{1.s.} \cdot dx \cdot dy \quad (15)$$

It follows that this is equivalent to replacing the residual in (11) by its least-squares fit over each element, or

$$\iint N_i \cdot R_{1.s.}^{(k)} \cdot dx \cdot dy = 0, \quad k = 1, 3; \quad i = 1, n. \quad (16)$$

The equivalence is only valid for rectangular elements. Consequently the use of reduced integration with triangular elements, which has been proposed by Zienkiewicz (1976), fails. However the direct application of (16), which is proposed as a new Method of Weighted Residuals by Fletcher (1977c), to triangular elements is successful as indicated by the results shown in Figure 2.

Thus certain alternative techniques for evaluating

(13) and (14) imply different formulations than (11) and these alternative techniques often produce superior final solutions.

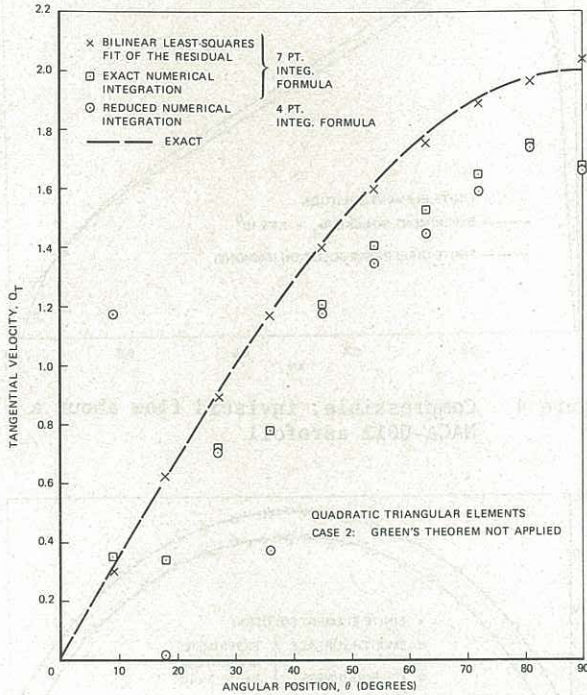


Figure 2 Comparison of reduced integration and least-squares residual fit for incompressible flow about a circular cylinder

4 SOLUTION OF THE NONLINEAR EQUATIONS

The algebraic relations, equations (12), can be written as functions of $\bar{\rho}_j$, $(\bar{\rho}u)_j$ and $(\bar{\rho}v)_j$ in the following form

$$\left. \begin{aligned} \sum_j a_{ij} \cdot (\bar{\rho}u)_j + \sum_j b_{ij} \cdot (\bar{\rho}v)_j &= 0 \\ \sum_j a_{ij} \left\{ \frac{(\bar{\rho}u)_j^2}{\bar{\rho}_j} + k \cdot \bar{\rho}_j^\gamma \right\} + \sum_j b_{ij} \cdot \frac{(\bar{\rho}u)_j \cdot (\bar{\rho}v)_j}{\bar{\rho}_j} &= 0 \\ \sum_j a_{ij} \frac{(\bar{\rho}u)_j \cdot (\bar{\rho}v)_j}{\bar{\rho}_j} + \sum_j b_{ij} \left\{ \frac{(\bar{\rho}v)_j^2}{\bar{\rho}_j} + k \cdot \bar{\rho}_j^\gamma \right\} &= 0 \end{aligned} \right\} \quad i=1, n \quad (17)$$

It is apparent that equations (17) define enough independent relationships to solve for the unknowns $\bar{\rho}_j$, $(\bar{\rho}u)_j$ and $(\bar{\rho}v)_j$. A solution to (17) has been sought using a generalised Newton's method.

If all the unknown nodal values are gathered together in a vector \bar{q} , then at the $(v+1)$ th step of an iterative process, a generalised Newton's method may be written

$$\bar{q}^{v+1} = \bar{q}^v - \lambda \cdot J_{(v)}^{-1} \cdot \bar{R}(\bar{q}^v) \quad (18)$$

In (18) the jacobian, $J = \partial \bar{R} / \partial \bar{q}$ where \bar{R} is a vector of all the equation residuals formed from the left hand side of (17). λ is a scalar; setting $\lambda = 1$ gives the conventional Newton's method.

A major problem with Newton's method is that J must be computed and inverted at each step of the iteration. The inversion of J requires large quantities of main storage and requires a large amount of computation time. Sparse matrix techniques have been

introduced (Fletcher, 1976) to economise on both main storage and on computation time. However other ways of making Newton's method more economical have also been utilized. Firstly J and J^{-1} have been held constant for a number of steps. Secondly trial solutions, \bar{q}_m^{v+1} , have been computed with different values of λ_m . The preferred solution, \bar{q}_p^{v+1} , is that which minimises s_m given by

$$s_m = \sum_{l=1}^n R_l^2 \quad (19)$$

These techniques have been effective and Newton's method has converged rapidly when close to the true solution. A typical set of results for compressible flow about a circular cylinder is shown in Figure 3.

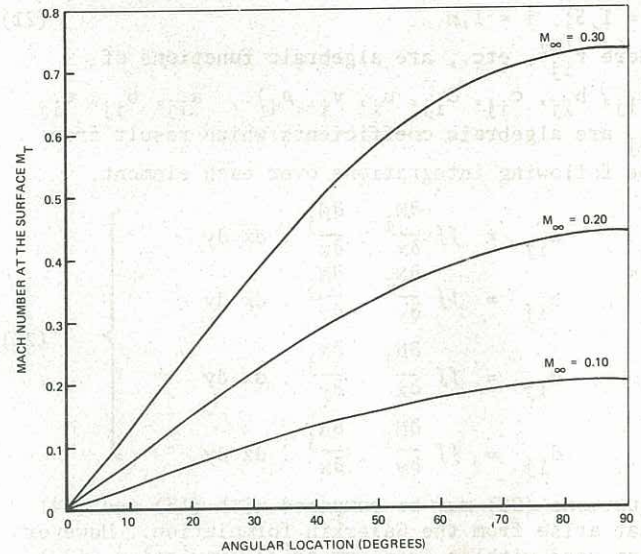


Figure 3 Compressible flow about a circular cylinder

Unfortunately as the grid is refined, i.e. the number of unknowns, n , is increased, the radius of convergence shrinks (Rheinboldt, 1974). This feature of Newton's method has been found to negate, to a considerable degree, the advantage of quadratic convergence when close to the true solution. For the flow about a circular cylinder it was not found possible to obtain convergence of Newton's method for Mach numbers above 0.32.

Successive over-relaxation (SOR) iterative techniques converge more slowly than Newton's method but are guaranteed to converge from any starting values. In an effort to utilize SOR iterative techniques the Galerkin formulation has been superseded by a least-squares formulation. This is described in the next section.

5 LEAST-SQUARES FORMULATION

The starting point for this formulation are the expressions for the residuals, equations (8) to (10). The least-squares formulation requires that

$$\iint (a_1 \cdot R^{(1)2} + a_2 \cdot R^{(2)2} + a_3 \cdot R^{(3)2}) \cdot dx \cdot dy = \text{minimum}, \quad (19)$$

where a_1 , a_2 and a_3 are scalars that may be used to adjust the significance of the various equations. Differentiating (19) with respect to each of the unknown nodal values in turn produces the result

$$\iint \left\{ a_1 \cdot \frac{\partial R^{(1)}}{\partial \bar{q}_i} \cdot R^{(1)} + a_2 \cdot \frac{\partial R^{(2)}}{\partial \bar{q}_i} \cdot R^{(2)} + a_3 \cdot \frac{\partial R^{(3)}}{\partial \bar{q}_i} \cdot R^{(3)} \right\} dx dy = 0, i = 1, n \quad (20)$$

where $\bar{q}_i = \{(\bar{\rho}u)_i, (\bar{\rho}v)_i, \bar{\rho}_i\}$. Substitution of equations (8) to (10) into (20), and evaluation of the integrals, produces the following algebraic equations.

$$S_i^{(m)} = \sum_j r_{ij}^{(m)} \cdot (\bar{\rho}u)_j + s_{ij}^{(m)} \cdot (\bar{\rho}v)_j + t_{ij}^{(m)} \cdot (\bar{\rho}u^2)_j + x_{ij}^{(m)} \cdot (\bar{\rho}uv)_j + y_{ij}^{(m)} \cdot (\bar{\rho}v^2)_j + z_{ij}^{(m)} \cdot \bar{p}_j = 0$$

$m = 1, 3; i = 1, n$ (21)

where $r_{ij}^{(m)}$, etc., are algebraic functions of $(a_{ij}, b_{ij}, c_{ij}, d_{ij}, u_i, v_i, \rho_i)$. $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are algebraic coefficients which result from the following integrations over each element.

$$\left. \begin{aligned} a_{ij} &= \iint \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} \cdot dx dy \\ b_{ij} &= \iint \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial y} \cdot dx dy \\ c_{ij} &= \iint \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial y} \cdot dx dy \\ d_{ij} &= \iint \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial x} \cdot dx dy \end{aligned} \right\} \quad (22)$$

Equations (22) may be compared with (13) and (14) that arise from the Galerkin formulation. However when used with an isoparametric formulation, (22) are considerably more complicated to evaluate than (13) and (14).

The iterative solution of (21) for $(\bar{\rho}u)_j, (\bar{\rho}v)_j$ and \bar{p}_j has been obtained from repeated applications of the following formula

$$q_i^{v+1} = q_i^v - \lambda \left[\frac{\partial S_i^{(m)}}{\partial q_i} (q_o) \right]^{-1} S_i^{(m)}(q_o) \quad (23)$$

In contrast to Newton's method $\partial S_i^{(m)} / \partial q_i$ is a scalar and trivial to invert, thus no excessive demand is made on storage or computation time. λ is a scalar and may be used to increase the rate of convergence.

The application of the least-squares formulation to compressible, inviscid flow about two representative aerofoils is indicated by Figures 4 and 5. The pressure distributions shown in Figure 4 have been obtained from the flow about a NACA-0012 aerofoil at zero angle of attack and a freestream Mach number of 0.40. For comparison experimental results (Amick, 1950) and finite difference computations (Emmons, 1948) are included.

The pressure distributions shown in Figure 5 have been obtained from the flow about a 6% circular-arc aerofoil at zero angle of attack and a freestream Mach number of 0.71. Experimental results (Knechtel, 1959) are shown for comparison.

The agreement between the finite element solutions and the experimental results is slightly misleading. Since the computational results do not allow

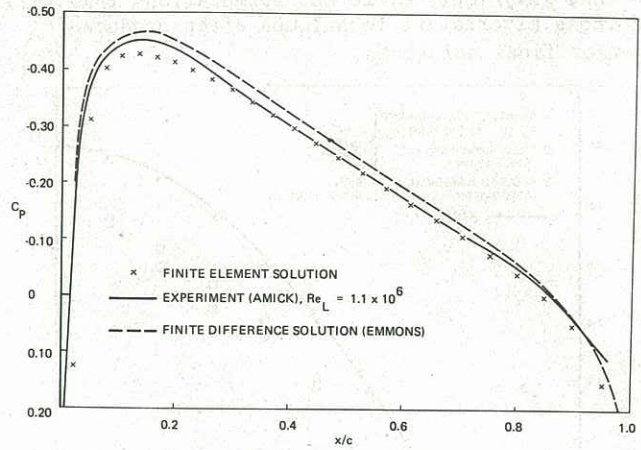


Figure 4 Compressible, inviscid flow about a NACA-0012 aerofoil

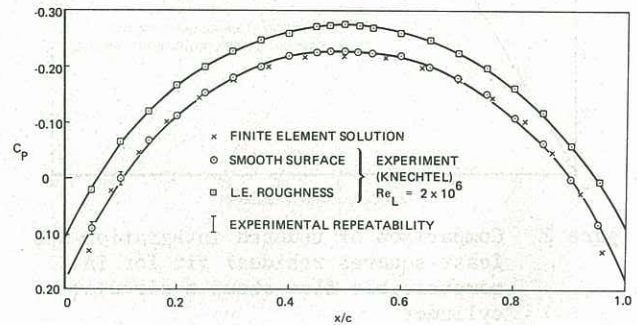


Figure 5 Compressible, inviscid flow about a 6% circular-arc aerofoil

for the displacement thickness the computational results would be expected to produce a pressure distribution that was more negative than that produced by the experiment with a smooth body surface. The failure of the computational results, shown in Figures 4 and 5, to produce a sufficiently negative pressure distribution is believed to be due mainly to applying the free-stream boundary conditions not sufficiently far from the body. Although the results presented are for purely sub-critical flows, finite element solutions exhibiting local supersonic flow, but without internal shocks, have been obtained for other configurations.

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