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Minimisation of Relaxation Drag

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SUMMARY Thermodynamic relaxation processes, set up in a fluid by a moving body, are a source of entropy production which causes a drag force on the body. For a slender two-dimensional body moving with uniform velocity this drag force is explicitly given as a functional of the velocity potential for the limiting cases of near-equilibrium and near-frozen flow. The result for near-equilibrium flow is used to calculate the shape of a profile which, for given area and given lift coefficient, has minimum relaxation drag.

1 DRAG CAUSED BY A RELAXATION PROCESS

Rate processes in a fluid, for example chemical reactions or relaxation of internal molecular degrees of freedom etc., are a source of entropy production. If these relaxation processes are due to changes in the thermodynamic state of the fluid which are caused by a body moving through the fluid, then as a consequence of the entropy production in the fluid the body experiences a drag force. The relation between entropy production and drag is easy to derive for a slender two-dimensional body (profile) moving with constant velocity u_∞ through the fluid. In a body-fixed frame of reference the flow is steady; the fluid approaches the body with velocity u_∞. The approaching fluid is in a thermodynamic equilibrium state. We assume subsonic flow throughout, such that no shock waves appear in the flow field.

In the vicinity of the body pressure and density, and hence the thermodynamic state of the fluid, are changed as compared with the upstream equilibrium state. Thereby relaxation processes are set up which generate entropy. Assuming the flow to be inviscid, these are the only entropy sources in the flow field. Because the disturbance created by the body decreases with increasing distance from the body, the entropy production will be higher on streamlines passing through the immediate vicinity of the body than on streamlines farther away. Therefore, an entropy wake emerges downstream, as shown in the figure. Far downstream of the body the flow is again a parallel flow and the fluid is in thermodynamic equilibrium. Consequently, Crocco's vorticity theorem is applicable (Becker, 1970):

$$-\underset{\sim}{v} \times \operatorname{curl} \underset{\sim}{v} = T \nabla s \qquad (1)$$

Here $\underset{\sim}{v}$ is the velocity, T the absolute temperature, and s the specific entropy. As (1) shows, the downstream flow possesses vorticity which is generated by the rate processes. The direction of the vortex vector is such that the velocity is decreasing with increasing entropy. Hence, the downstream velocity profile exhibits a wake with diminished velocity (see the figure).

The drag force D on the body (per unit length perpendicular to the flow plane) is

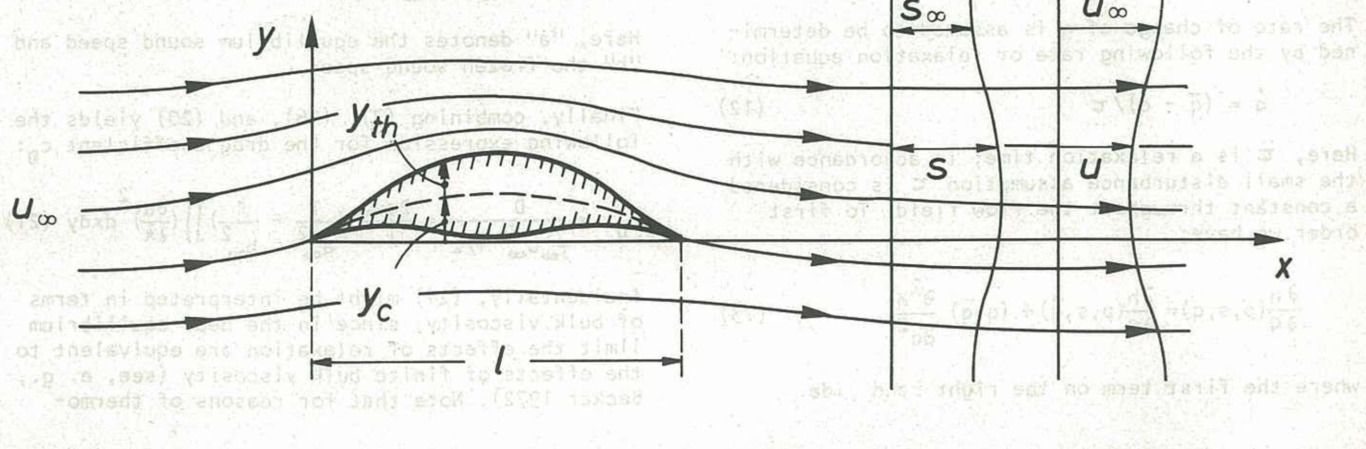
$$D = \int \rho u (u_\infty - u)\, dy \qquad (2)$$

The integration has to be performed on a line $x = \text{const}$ so far downstream that pressure disturbances created by the body have disappeared there. For a slender profile the disturbance of the parallel flow, and hence also the entropy production, remains small, so that $|u_\infty - u| \ll u_\infty$. Assuming slenderness of the profile we linearize the theory with respect to disturbances of the undisturbed parallel equilibrium flow. Equation (2) is thereby transformed into:

$$D = \rho_\infty u_\infty \int (u_\infty - u)\, dy \qquad (3)$$

The linearized form of Crocco's vorticity theorem is

$$u_\infty \frac{\partial u}{\partial y} = - T_\infty \frac{\partial s}{\partial y} \qquad (4)$$



On integration (4) yields:

$$u_\infty (u_\infty - u) = T_\infty (s - s_\infty) \qquad (5)$$

Combination of (3) and (5) leads to

$$Du_\infty = T_\infty \rho_\infty u_\infty \int (s - s_\infty)\, dy \qquad (6)$$

Now, $\rho_\infty u_\infty \int (s - s_\infty)\, dy$ is the surplus of entropy leaving a large control volume surrounding the body. Therefore, (6) may be written as

$$D = T_\infty \dot{S}/u_\infty = \frac{T_\infty}{u_\infty} \iint \sigma \, dx\, dy \qquad (7)$$

Here, $\dot{S}$ is the entropy generated per unit time in the flow field, and σ is the entropy generated per unit time and unit volume of the flow field. The integration in (7) extends over the whole flow field. Equation (7) generalizes a result originally derived by Oswatitsch (1945) to flows with relaxation. Though here derived for a slender profile, it is of quite general validity (cf. Romberg 1966, Becker 1970).

2 ENTROPY PRODUCTION

For evaluating the entropy production σ in (7) we assume that only one rate process has to be taken account of which necessitates the introduction of one internal state variable q. (The generalisation to more than one rate process is rather straightforward, but does not yield additional insight). A possible form of the canonical equation of state of the fluid is

$$h = h\,(p, s, q) \qquad (8)$$

Here, h is the specific enthalpy, p the pressure. The differentials of the four state variables in (8) are connected through Gibbs' relation:

$$dh = T\, ds + \frac{1}{\rho}\, dp + \frac{\partial h}{\partial q}\, dq \qquad (9)$$

Since for inviscid flow $\dot{h} = \dot{p}/\rho$, with the dot denoting the material derivative with respect to time, (9) immediately yields:

$$\dot{s} = \frac{\sigma}{\rho} = - \frac{1}{T} \frac{\partial h}{\partial q} \dot{q} \qquad (10)$$

For thermodynamik equilibrium the internal state variable q is a function $\bar{q}(p, s)$ of pressure and entropy, which is implicity given by

$$\frac{\partial h}{\partial q} (p, s, \bar{q}\,(p, s)) = 0 \qquad (11)$$

The rate of change of q is assumed to be determined by the following rate or relaxation equation:

$$\dot{q} = (\bar{q} - q)/\tau \qquad (12)$$

Here, τ is a relaxation time; in accordance with the small disturbance assumption τ is considered a constant throughout the flow field. To first order we have:

$$\frac{\partial h}{\partial q}(p,s,q) = \frac{\partial h}{\partial q}(p,s,\bar{q}) + (q-\bar{q}) \frac{\partial^2 h}{\partial q^2}, \qquad (13)$$

where the first term on the right hand side vanishes because of (11). By combining (10), (12) and (13) one derives

$$\sigma = \frac{\rho_\infty}{T_\infty} \left(\frac{\partial^2 h}{\partial q^2} \right)_\infty \frac{(\bar{q} - q)^2}{\tau} \qquad (14)$$

The index "∞" indicates that the corresponding quantity has to be taken at its upstream value in agreement with the small disturbance linearization.

3 DRAG FOR NEAR-EQUILIBRIUM AND FOR NEAR-FROZEN FLOW

The entropy production as given by (14) can be related in a simple way to the velocity field for two limiting cases:

Case a: Near-equilibrium flow. In that case the relaxation time τ is so small compared with the characteristic flow time l/u_∞ (with l the profile length) that everywhere $q \approx \bar{q}$, $\dot{q} \approx \dot{\bar{q}}$. Therefore, (12) yields:

$$(\bar{q} - q)^2/\tau \approx \tau \dot{\bar{q}}^2 \qquad (15)$$

Inserting (15) into (14) one obtains

$$\sigma = \frac{\rho_\infty}{T_\infty} \left(\frac{\partial^2 h}{\partial q^2}\right)_\infty \tau \dot{\bar{q}}^2 \qquad (16)$$

Now,

$$\dot{\bar{q}} = \frac{\partial \bar{q}}{\partial p} \dot{p} + \frac{\partial \bar{q}}{\partial s} \dot{s} \qquad (17)$$

Because the entropy production is a second order effect (cf. (14)) we may neglect the second term on the right hand side of (17). From (11) we derive

$$\frac{\partial \bar{q}}{\partial p} = - \frac{\partial^2 h}{\partial p \partial q} \Big/ \frac{\partial^2 h}{\partial q^2} \qquad (18)$$

Combination of (17) and (18), and taking account of $\dot{p} = u_\infty\, \partial p/\partial x$ leads to:

$$\dot{\bar{q}}^2 = \left(\frac{\partial^2 h}{\partial p \partial q} \Big/ \frac{\partial^2 h}{\partial q^2}\right)^2 \cdot \left(u_\infty \frac{\partial p}{\partial x}\right)^2 \qquad (19)$$

Using formula (3.3) from Becker (1972) and the linearized Euler's equation $\rho_\infty u_\infty\, \partial u/\partial x = -\partial p/\partial x$, we may recast (19) into the form:

$$\dot{\bar{q}}^2 = \left(\frac{1}{a_\infty^2} - \frac{1}{b_\infty^2}\right) \left(\frac{\partial^2 h}{\partial q^2}\right)_\infty^{-1} \left(u_\infty^2 \frac{\partial u}{\partial x}\right)^2 \qquad (20)$$

Here, "a" denotes the equilibrium sound speed and "b" the frozen sound speed.

Finally, combining (7), (16), and (20) yields the following expression for the drag coefficient c_D:

$$c_D := \frac{D}{\rho_\infty u_\infty^2\, 1/2} = \frac{2\tau u_\infty}{1} \left(\frac{1}{a_\infty^2} - \frac{1}{b_\infty^2}\right) \iint \left(\frac{\partial u}{\partial x}\right)^2 dx dy \qquad (21)$$

Incidentally, (21) might be interpreted in terms of bulk viscosity, since in the near-equilibrium limit the effects of relaxation are equivalent to the effects of finite bulk viscosity (see, e. g., Becker 1972). Note that for reasons of thermodynamic stability a_∞ is always smaller than b_∞ ; consequently c_D is always positive, as it should be. A result equivalent to (21) has first been derived, under the severe restriction to vibrational relaxation of a perfect gas by Romberg (1967).

Case b: Near-frozen flow. Here $\tau \gg 1/u_\infty$. Therefore, the value of q nowhere deviates appreciably from the upstream equilibrium value $q(p_\infty, s_\infty)$. Now,

$$\bar{q}(p,s)=\bar{q}(p_\infty,s_\infty)+\frac{\partial\bar{q}}{\partial p}(p-p_\infty)+\frac{\partial\bar{q}}{\partial s}(s-s_\infty) \qquad (22)$$

The third term on the right hand side is again neglected as a second order effect. Hence, using (18), we obtain with $\bar{q}(p_\infty, s_\infty) \approx q$:

$$\bar{q}(p,s) - q = -\left(\frac{\partial^2 h}{\partial p\partial q} / \frac{\partial^2 h}{\partial q^2}\right)_\infty (p - p_\infty) \qquad (23)$$

The linearized Euler's equation yields $p - p_\infty = -\rho_\infty u_\infty (u - u_\infty)$. Using this and formula (3.3) from Becker (1972) we derive:

$$(\bar{q}-q)^2 = \left(\frac{1}{a_\infty^2} - \frac{1}{b_\infty^2}\right)\left(\frac{\partial^2 h}{\partial q^2}\right)_\infty^{-1} u_\infty^2 (u - u_\infty)^2 \qquad (24)$$

Combining (7), (14), and (24) yields the following value for the drag coefficient c_D:

$$c_D = \frac{2l}{\tau u_\infty}\left(\frac{1}{a_\infty^2} - \frac{1}{b_\infty^2}\right)\frac{1}{l^2}\iint (u-u_\infty)^2 dx\, dy \qquad (25)$$

4 DETERMINATION OF THE FLOWFIELD

In order to evaluate the integrals in (21) and (25) for a given profile one has to solve the linearized flow equations for equilibrium and frozen flow respectively. For subsonic flow a velocity potential φ' exists which satisfies the equation

$$(1 - M_\infty^2)\frac{\partial^2\varphi'}{\partial x^2} + \frac{\partial^2\varphi'}{\partial y^2} = 0 \qquad (26)$$

The velocity components u, v are given by

$$u - u_\infty = \frac{\partial\varphi'}{\partial x}, \quad v = \frac{\partial\varphi'}{\partial y} \qquad (27)$$

In (26) M_∞ has to be chosen as the equilibrium Machnumber u_∞/a_∞ in case a and as the frozen Machnumber u_∞/b_∞ in case b. The profile shape is given by (see figure):

$$\begin{aligned}&\text{upper side: } y_u(x) = y_c(x) + y_{th}(x)\\ &\text{lower side: } y_l(x) = y_c(x) - y_{th}(x)\end{aligned} \qquad (28)$$

The centerline of the profile is given by y_c, and $2y_{th}$ is the thickness of the profile. Apart from $\varphi' \to 0$ for $x^2 + y^2 \to \infty$, the potential has to satisfy the following boundary conditions on the x-axis between x = o and x = 1:

$$\frac{\partial\varphi'}{\partial y} = u_\infty\frac{dy_u}{dx} = u_\infty\frac{dy_c}{dx} + u_\infty\frac{dy_{th}}{dx} \quad \text{for } y = +0 \qquad (29a)$$

$$\frac{\partial\varphi'}{\partial y} = u_\infty\frac{dy_l}{dx} = u_\infty\frac{dy_c}{dx} - u_\infty\frac{dy_{th}}{dx} \quad \text{for } y = -0 \qquad (29b)$$

It is convenient to split the potential into two parts:

$$\varphi' = \varphi'_c + \varphi'_{th} \qquad (30)$$

satisfying the following boundary conditions:

$$\frac{\partial\varphi'_c}{\partial y} = u_\infty\frac{dy_c}{dx} \quad \text{for } y = \pm 0, \quad 0 \le x \le 1 \qquad (31)$$

and

$$\frac{\partial\varphi'_{th}}{\partial x} = \pm u_\infty\frac{dy_{th}}{dx} \quad \text{for } y = \pm 0, \ 0 \le x \le 1 \qquad (32)$$

The potential φ'_c determines the lift of the profile, while φ'_{th} takes account of the displacement effect due to finite thickness of the profile.

It is obvious that φ'_c together with $\partial\varphi'_c/\partial x$ and $\partial^2\varphi'_c/\partial x^2$ change sign when y_c is changed into $-y_c$, whereas φ'_{th} is unaffected thereby. This change means a mirror reflection of the profile, and hence also of the flow field, at the x-axis. Such a reflection leaves the total entropy production and, therefore, also the drag unchanged. In case "a" the drag is proportional to

$$\iint\left(\frac{\partial^2\varphi'_c}{\partial x^2} + \frac{\partial^2\varphi'_{th}}{x^2}\right)^2 dx\, dy = \qquad (33)$$

$$\iint\left(\frac{\partial^2\varphi'_c}{\partial x^2}\right)^2 dxdy + \iint\left(\frac{\partial^2\varphi'_{th}}{\partial x^2}\right)^2 dxdy + 2\iint\frac{\partial^2\varphi'_c}{\partial x^2}\frac{\partial^2\varphi'_{th}}{\partial x^2}dxdy$$

The last term in (33) changes sign under reflection. Because the drag is unaffected by reflection, this term must vanish. Therefore, the total drag is the sum of a drag component due to y_c and a component due to y_{th} . The same obviously holds true for case "b". In other words: Finite thickness and lift both contribute to the drag; the two contributions are additive.

Application of the Prandtl-Glauert-transformation:

$$\varphi'(x,y) = \frac{1}{\sqrt{1 - M_\infty^2}}\varphi(x,\eta), \quad \eta = \sqrt{1 - M_\infty^2}\, y \qquad (34)$$

leads to the Laplace-equation:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial \eta^2} = 0 \qquad (35)$$

After splitting the transformed potential φ into $\varphi = \varphi_c + \varphi_{th}$ the boundary conditions can be written as follows:

$$\frac{\partial\varphi_c}{\partial\eta} = u_\infty\frac{dy_c}{dx}, \quad \frac{\partial\varphi_{th}}{\partial\eta} = \pm u_\infty\frac{dy_{th}}{dx} \quad \text{for } \eta = \pm o, \ o \le x \le 1 \qquad (36)$$

The drag formula (21) is transformed into

$$c_D = C \iint \left[\left(\frac{\partial^2 \varphi_{th}}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \varphi_c}{\partial x^2}\right)^2 \right] dx\, d\eta \quad (37)$$

with

$$C = \frac{2\tau u_\infty}{l} \left(\frac{1}{a_\infty^2} - \frac{1}{b_\infty^2}\right) (1 - M_\infty^2)^{-3/2} \quad (38)$$

The solution for φ_c and φ_{th} can be written as:

$$2\pi \varphi_{th}(x,\eta) = u_\infty l \int_0^1 q(\xi) \ln \sqrt{\left(\frac{x}{l} - \xi\right)^2 + \left(\frac{\eta}{l}\right)^2}\, d\xi \quad (39)$$

$$2\pi \varphi_c(x,\eta) = u_\infty l \int_0^1 \gamma(\xi) \operatorname{arctg} \frac{\eta/l}{\frac{x}{l} - \xi}\, d\xi \quad (40)$$

Here q and γ are a dimensionless source and vortex distribution on the x-axis. These distributions are determined by y_{th} and y_c respectively (cf. Weissinger 1963). The source distribution has to satisfy the "closing condition":

$$\int_0^1 q(\xi)\, d\xi = 0 \quad (41)$$

and the vortex distribution is subjected to the Kutta-Joukowski-condition (smooth flow at the trailing edge):

$$\gamma(1) = 0 \quad (42)$$

The expressions (39) and (40) have the following properties:

$$\frac{\partial \varphi_{th}}{\partial \eta} = \pm u_\infty \frac{q(x)}{2} \quad (43)$$

for $\eta = \pm 0,\ 0 \leq x \leq 1$

$$\frac{\partial \varphi_c}{\partial x} = \pm u_\infty \frac{\gamma(x)}{2} \quad (44)$$

The total area A enclosed by the profile is given by

$$A = -2 \int_0^1 x \frac{dy_{th}}{dx}\, dx = -l^2 \int_0^1 \xi\, q(\xi)\, d\xi \quad (45)$$

Finally, the lift coefficient c_L of the profile is given by

$$c_L = \frac{2}{u_\infty l} \int_0^1 \left(\left.\frac{\partial \varphi'}{\partial x}\right|_{y=+0} - \left.\frac{\partial \varphi'}{\partial x}\right|_{y=-0} \right) dx = \frac{2}{\sqrt{1 - M_\infty^2}} \int_0^1 \gamma(\xi)\, d\xi \quad (46)$$

5 PROFILE WITH MINIMUM DRAG FOR NEAR-EQUILIBRIUM FLOW

Inserting the expressions (39) and (40) into (21) one obtains, after rather lengthy mathematical derivations, the result:

$$\frac{c_D}{C} = -\frac{1}{4\pi} \int_0^1 q(\zeta)\, d\zeta \frac{d}{d\zeta} \oint_0^1 \frac{q(\xi)\, d\xi}{\xi - \zeta} - \frac{1}{4\pi} \int_0^1 \gamma(\zeta)\, d\zeta \frac{d}{d\zeta} \oint_0^1 \frac{\gamma(\xi)\, d\xi}{\xi - \zeta} \quad (47)$$

Here, $\oint$ denotes the Cauchy principal value of the integral.

(47) is the starting point for finding the shape of a profile with given area A and given lift-coefficient c_L which has minimum drag coefficient c_D: The variation of c_D/C, with (41), (45), (46) as auxiliary conditions taken account of by three Lagrangian multipliers μ, λ, ν, is:

$$\delta\left(\frac{c_D}{C}\right) = -\frac{1}{2\pi} \int_0^1 \delta q(\zeta)\, d\zeta \left[\frac{d}{d\zeta} \oint_0^1 \frac{q(\xi)}{\xi - \zeta}\, d\xi - \lambda\zeta - \mu\right] - \frac{1}{2\pi} \int_0^1 \delta\gamma(\zeta)\, d\zeta \left[\frac{d}{d\zeta} \oint_0^1 \frac{\gamma(\xi)}{\xi - \zeta}\, d\xi - \nu\right] \quad (48)$$

The derivation of (48) from (47) is valid only if q and γ are differentiable and satisfy the conditions $q(1) = \gamma(1) = q(o) = \gamma(o) = o$; that the solution of the optimum problem satisfies these conditions can be confirmed a posteriori (cf. (51), (52)).

For δc_D to vanish, the following equations have to be satisfied:

$$\frac{d}{d\zeta} \oint_0^1 \frac{q(\xi)}{\xi - \zeta}\, d\xi = \lambda\zeta + \mu \quad (49)$$

$$\frac{d}{d\zeta} \oint_0^1 \frac{\gamma(\xi)}{\xi - \zeta}\, d\xi = \nu \quad (50)$$

The solution of these singular integral equations is well-known from classical aerodynamics (Weissinger 1963). Taking account of (41), (42), (45), (46) we can write the result as:

$$q\left(\frac{x}{l}\right) = \frac{2^5}{\pi} \frac{A}{l^2} \left(1 - \frac{2x}{l}\right) \sqrt{\frac{x}{l}\left(1 - \frac{x}{l}\right)} \quad (51)$$

$$\gamma\left(\frac{x}{l}\right) = \frac{2^2}{\pi} c_L \sqrt{1 - M_\infty^2} \sqrt{\frac{x}{l}\left(1 - \frac{x}{l}\right)} \quad (52)$$

The corresponding profile thickness and centerline are:

$$y_{th}(x) = \frac{2^6}{3\pi} \frac{A}{l} \left[\frac{x}{l}\left(1 - \frac{x}{l}\right)\right]^{3/2} \quad (53)$$

$$y_c(x) = \frac{1}{\pi} c_L \sqrt{1 - M_\infty^2}\, \frac{x}{l}\left(1 - \frac{x}{l}\right) \quad (54)$$

Finally, the two contributions to the drag coefficient are:

$$c_{D,th} = \frac{16}{\pi} C (A/l)^2 \quad (55)$$

$$c_{D,c} = \frac{1}{2\pi} C c_L^2 (1 - M_\infty^2) \quad (56)$$

with C given by (38).

It is to be noted that the minimum drag profile thus found is symmetrical with respect to x = 1/2 and has a sharp leading and trailing edge. The calculated equilibrium flow field is also symmetrical with respect to x = 1/2, and the flow is therefore smooth not only at the trailing but also at the leading edge. The assumptions of the small disturbance theory are therefore satisfied: for slender profiles the disturbances of the parallel flow remain small in all points of the flow field. Flow round an edge leads to infinite values of $\partial\varphi/\partial x$ at the edge with consequent divergence of the drag integral. Of course, in such a case the small perturbation theory is no longer applicable in the whole flow field. However, it is clear that flow round an edge would considerably increase the drag, as would also flow near the stagnation point at a rounded nose of the profile. Therefore, the optimum profile must have sharp leading and trailing edges with smooth flow at both edges. The profile shape is shown qualitatively in the figure.

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