Stratified Flow Over a Long Obstacle: Theory vs. Experiment

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ABSTRACT   This paper describes an experimental study of the flow of stratified fluid over a long slender obstacle in a channel when the flow becomes sub-critical with respect to the first internal wave mode. A general description of the nature of the flow field is given, and various properties are compared quantitatively with predictions from the linearized-boundary-condition theory and with a quasi-linear extension of this theory by the author. Some measure of agreement with these theories is obtained, provided that the flow is not too close to criticality for any mode; in the latter case, non-linear effects of a hydraulic character become important.

1   INTRODUCTION

There are several theories which aim to describe the flow of stratified fluids over two-dimensional obstacles. Of these, the most commonly used (particularly in a meteorological context) is the linear-ized-boundary-condition (LBC) theory, which is applied to obstacles of small height (e.g. Brether-ton 1969, McIntyre 1972). A second type of theory is the kind due to Long (1955) and Yih (1960) which may be applied to obstacles of arbitrary height but require assumptions about the flow properties far upstream. This type of model has been compared qualitatively with observations by Long (1955) with reasonable success, but a more detailed and quant-itative comparison by Baines (1977a) shows that it is not applicable in general, because the motion of the obstacle (relative to the fluid) generates motions which propagate far upstream and hence violate the implicit assumption of no upstream influence. Also, the only quantitative experiment (to my knowledge) which has been carried out to test the LBC theory is that by Smith (1976), who found, in both the atmosphere and the laboratory for a two-layer configuration, that the theory predicted the lee wave-length correctly but under-estimated the amplitude by a factor of four.

In this paper quantitative observations of flow over a long slender semi-elliptical obstacle are compared with corresponding predictions from LBC theory and with those from a quasi-linear theory by the author (Baines 1977b). The latter may be regarded as an extension of the LBC theory which takes account of the obstacle's finite height by using the concept of a travelling forcing field.

2   LINEAR AND QUASI-LINEAR THEORIES

With horizontal (x) and vertical (z) axes fixed relative to the topography the dynamical equations, linearized by using the Oseen approximation, become

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \right)^2 \nabla^2 \psi + N^2 \psi_{xx} = 0 \quad , \tag{1}$$

where

$$u = -\psi_z, \quad w = \psi_x, \tag{2}$$

N is the Brunt-Väisälä frequency and U is the velocity of the obstacle through the fluid. The notation is standard, and a list of symbols is given in the appendix. Boundary conditions are

$$\left. \begin{array}{ll} w = U\frac{dh(x)}{dx} & \text{on} \quad z = h(x) \ , \\[2mm] w = o & \text{on} \quad z = D \ , \end{array} \right\} \tag{3}$$

for flow in a channel of depth D. The conventional linear theory then replaces the first boundary condition by

$$w = U\frac{dh(x)}{dx} \quad \text{on } z = o \ . \tag{4}$$

This approach has been formalized by McIntyre (1972) as a perturbation expansion in powers of small obstacle height, and yields no motion far upstream at second (and probably all) orders, and lee waves downstream. For the experimental situation studied here, with a semi-elliptical obstacle of height h and length 2a, in the limit of large time this theory gives for the stream function

$$\psi = o \qquad \qquad \text{upstream} \ ,$$

$$= 2Uh \sum_{n=1}^{n_K} n\, J_1\!\left[\frac{(1 - n^2/K^2)^{\frac{1}{2}} \frac{Na}{U}}{K^2 - n^2} \right] \cdot \sin(K^2 - n^2)^{\frac{1}{2}} \frac{\pi x}{D} \cdot$$

$$\sin\frac{n\pi z}{D} + O(h^2) \quad \text{downstream} \ , \tag{5}$$

where n_K is the largest integer less than $K = \frac{ND}{\pi U}$. The various terms in the series represent the lee-wave modes.

It has been shown by the author that the above formalism is not a sound procedure, because the transfer of the lower boundary condition from z=h(x) to the horizontal surface z = o changes the mathematical character of the problem from a well-posed one for the 4th order partial d.e. to one in which the boundary condition is specified on a characteristic, and that consequently part of the solution is lost (including, in particular, the upstream part). Further, this part cannot be regained by an expansion to powers of any order in h if the boundary condition is always applied at z = o.

In an attempt to surmount this difficulty and incor-porate the finite height of the obstacle (however small) the author (ibid) has constructed a "quasi-

linear" model based on a travelling forcing field caused by potential flow over the obstacle, which derives from the transient solution of Bretherton (1967). Approximations made essentially involve the neglect of non-linear effects in the body of the fluid. This solution is for infinite depth and reduces the potential flow as $N \to 0$ and to the LBC solution as $h \to 0$. It also reduces to Bretherton's (1967) slow flow solution as $U \to 0$, but its region of applicability is probably restricted to $\frac{Nh}{U}$ being small.

The corresponding solution for the case of finite depth may be obtained from the solution for infinite depth by regarding the upper rigid surface as a reflecting surface, and making the additional assumption that the effect of the finite size of the obstacle on waves reflecting from the lower surface may be neglected. On this basis, the coefficients of the various modes making up the finite depth solution may be obtained from the Fourier transform (w.r.t.z)of the infinite depth solution. The solution in the limit of large time is

$$= Uh \sum_{n=1}^{n_K} \frac{J_1\left(\frac{n\pi h}{D}\right)}{1 - \frac{n}{K}} \sin \frac{n\pi z}{D} \quad , \text{ upstream} \quad (6)$$

$$= 2Uh \sum_{n=1}^{n_K} \frac{n}{K(K^2-n^2)^{\frac{1}{2}}} \frac{J_1\left(\frac{NH}{U}\right)}{H} \left[\frac{nh}{(K^2-n^2)^{\frac{1}{2}}} \cos \frac{Nx}{U}\left(1-\frac{n^2}{K^2}\right)^{\frac{1}{2}} \right.$$

$$\left. -a \sin \frac{Nx}{U}\left(1-\frac{n^2}{K^2}\right)^{\frac{1}{2}} \right] \sin \frac{n\pi z}{D}$$

$$+ Uh \sum_{n=1}^{n_K} \left[\frac{J_1\left(\frac{n\pi h}{D}\right) - J_1(R)}{1-\frac{n}{K}} - \frac{J_1(R)}{1+\frac{n}{K}} \right] \sin \frac{n\pi z}{D}$$

$$- Uh \sum_{n=n_K+1}^{\infty} \frac{2J_1(R)}{1-n^2/K^2} \sin \frac{n\pi z}{D} \quad , \text{ downstream}$$

where

$$R= \frac{Nh}{U} \text{ and } H^2 = h^2\frac{n^2}{K^2} + a^2\left(1-\frac{n^2}{K^2}\right) \quad . \quad (7)$$

Note that, if h is small, the upstream terms are $O(h^2)$ and that the downstream terms are identical with those of equation(5) to first order. Also, the solution becomes singular as K approaches an integer (as does equation (5), although less dramatically). This will be commented on later, but it clearly indicates that these theories are not applicable if K is too close to an integer value, in which case non-linear effects in the fluid must be important.

3   EXPERIMENTAL APPARATUS

A schematic plan of the apparatus giving relevant dimensions is shown in figure 1. The tank was filled to a depth of 34.0, 19.0 or 9.3 cm with a stratified fluid of constant Brunt-Väisälä frequency. Flow visualization was by means of neutrally buoyant beads, and the tank was illuminated from below through the transparent bottom by fluorescent lights which were collimated so as to illuminate the central region (spanwise) of the tank. The flow field immediately over the obstacle was illuminated by a fluorescent light overhead which travelled along the tank with the obstacle.
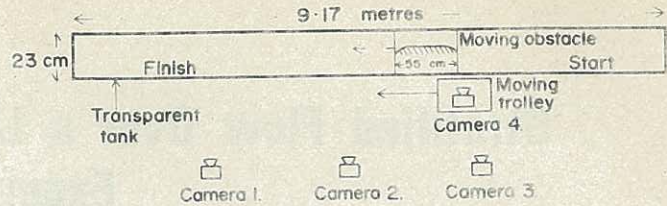

Figure 1 Schematic plan of experimental layout

The semi-elliptical obstacle was towed at a constant speed starting from rest from right to left along the tank, and data were recorded by means of streak photographs taken by four cameras - three at fixed stationary positions down the tank, and a fourth on a moving trolley which travelled along with the obstacle, recording the flow field immediately behind it. The three stationary cameras gave a check on the constancy of the towing speed, showed the development of the flow as the obstacle travelled down the tank, and provided a clear picture of the flow around the obstacle. The fourth (moving) camera gave measurements of the wavelength, amplitude and phase of the lee waves. Observations were concentrated on the deepest of the three depths mentioned above, since the primary aim was to test theories based on small obstacle height.

4   OBSERVATIONS AND COMPARISON WITH THEORY

The structure of the steady-state flow patterns (as attained in the channel) in the immediate vicinity of the obstacle as seen by the stationary cameras is shown in figure 2 for increasing values of K for D = 34.0 cm ($\frac{h}{D}$ = 0.076), the deepest of the 3 depths mentioned in the previous section.   For
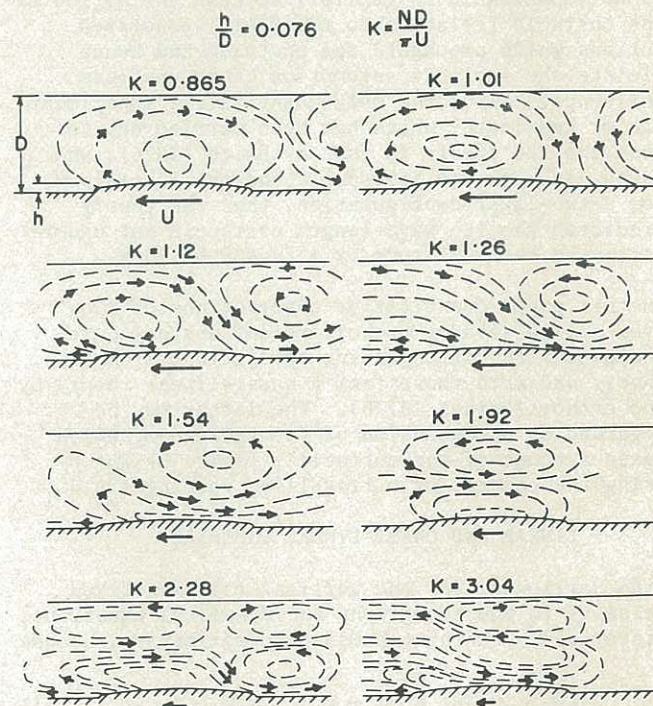

Figure 2 Streamline flow patterns over a moving obstacle. The patterns were apparently steady except for the motion extending on the upstream side for K = 1.01, 1.12, 2.28. Arrow lengths indicate relative velocities

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D = 19.0 cm the general flow patterns were similar for corresponding values of K, but for D = 9.3 cm the flow was clearly in a different regime with $\frac{h}{D}$ too large for a linear theory to be relevant. This shallow case will not be discussed further here.

For the supercritical case K < 1 the flow is approximately symmetrical over the obstacle, resembling potential flow, but this is immediately followed by a region of descending motion behind the obstacle, forming an "eddy" whose length increases linearly with time. This "eddy" is terminated by a region of rising motion which travels at a speed which is a little slower than the speed of long internal waves ($\frac{ND}{\pi}$) of the first mode. Behind this region the only motions are small weak eddies with no discernible ordered structure. As discussed below, although this flow is supercritical it has a resemblance to the familiar sluice jump - hydraulic jump combination of subcritical open channel hydraulics.

As K increases from 1 to 2 the motion over the obstacle progressively changes from a mode 1 to a mode 2 structure with the second eddy initially on the lee side progressively surmounting and compressing the first. The front of the initial eddy progresses upstream with the long wave speed, but for K ≳ 1.5 it eventually cuts off and terminates, apparently due to non-linear effects associated with the second eddy partially compressing the first.

For the lee waves downstream, the theoretical predictions of the LBC and "quasi-linear" theories are virtually identical (apart from K∿ integer where neither is relevant) for the obstacle shape used ($\frac{h}{a}$ = 0.085). The theoretical curves are therefore drawn as one, and figure 3 shows the comparison with the experimentally observed wavelengths of mode 1; figure 4 compares the amplitudes and phase (phase ϕ is defined by $\psi = -A \sin (kx - \phi)$ where A is positive). Provided K is not near unity the theory and experiments exhibit a general (though imperfect) agreement for the properties of the lee waves.

Results for a similar comparison of motion on the upstream side are shown in figure 5, where the termination of the mode 1 upstream motions for K ≳ 1.5 is indicated (such termination is not observed for the higher order modes when K > 2). The agreement here with the quasi-linear theory is not
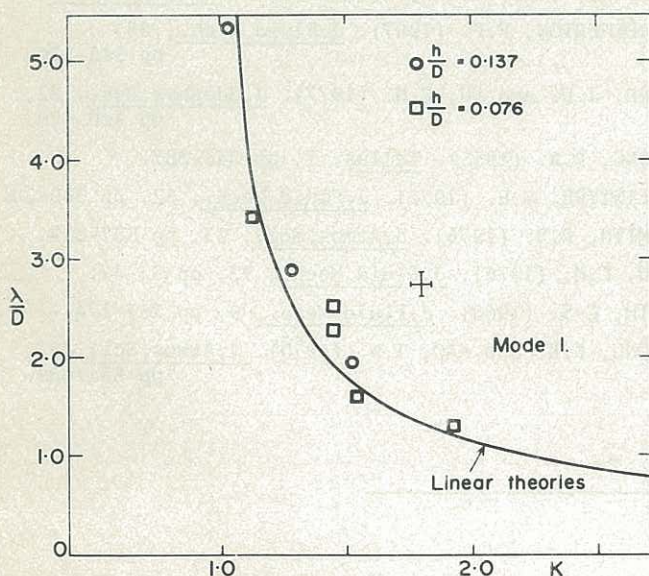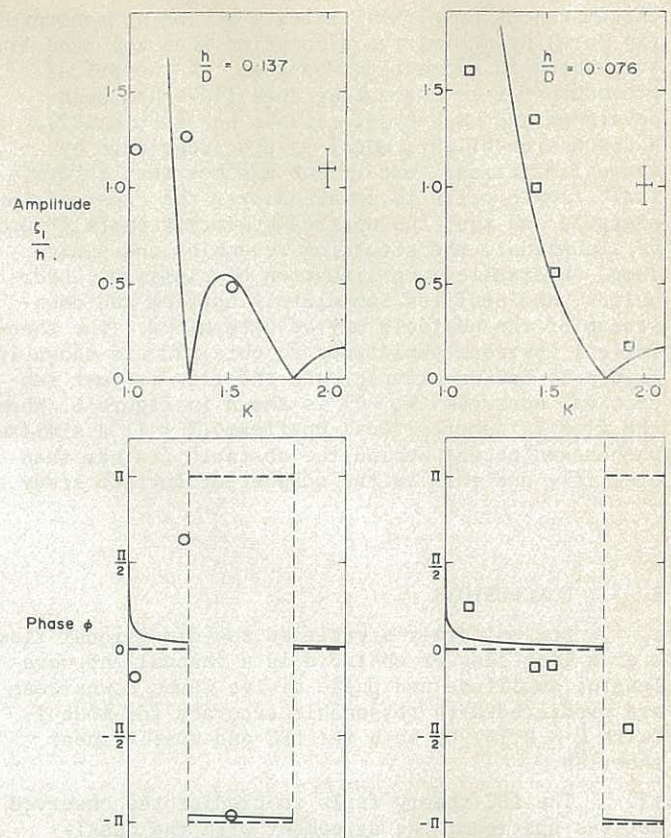
Figure 4 Amplitude and phase of lee waves. The dashed line denotes the phase according to LBC theory, the solid line according to the quasi-linear theory. The Bessel-function structure of the wave amplitude causes the phase to change by a factor of π when the former vanishes

as good as for the lee waves, but the general trend of the results is consistent, except for K near unity. Corresponding theoretical curves from the theory of Wong & Kao (1970) for a source-like semi-infinite obstacle are also shown. The stream function upstream from this model is given by

$$\psi = \frac{Uh}{\pi} \frac{1}{1-1/K} \sin \frac{\pi z}{D} , \qquad (8)$$

for the first mode, and the corresponding curves are also consistent with the experimental results.

Near K = 1 non-linear effects must become important, because of a resonance effect with the "columnar disturbance modes" of infinite wavelength. In these
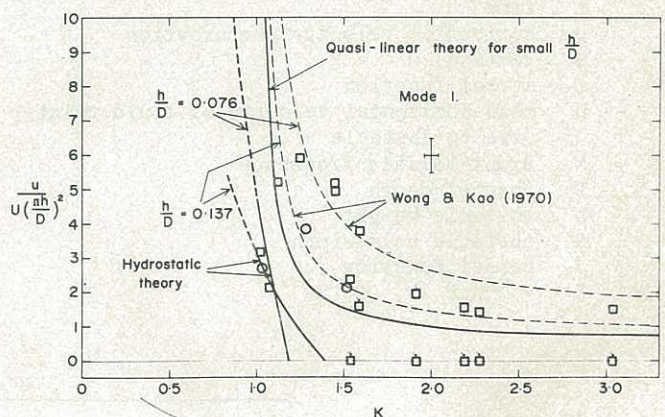
Figure 3 Wavelength/depth as a function of K = ND/πU

Figure 5 Upstream amplitudes of the maximum horisontal perturbation velocity measured near z = D (i.e. at the top of the channel), scaled with $U(\frac{\pi h}{D})^2$, as a function of K. See text

circumstances non-linear effects may be accommodated (in part) by making the assumption that the flow is everywhere in hydrostatic balance. A numerical procedure for investigating such flows has been developed and used by Su (1976), Lee & Su (1977). A discussion of this model is precluded here by space limitations, but if one assumes that a "critical" flow profile is attained over the crest of the obstacle and that the upstream velocity profile must be sinusoidal, the situation resembles the sluice jump - hydraulic jump situation of classical hydraulics, and profiles immediately upsteam and downstream of the obstacle may be determined. The theoretical upstream amplitudes so obtained are shown in figure 5, and an example when the flow is just subcritical upstream (K > 1) is shown in figure 6. When the flow is supercritical upstream (K < 1) a similar phenomenon occurs behind the obstacle (rather than over it), and this is the subject of further study.

5   CONCLUSIONS

i.   For uniformly stratified two-dimensional flow over a long slender obstacle in a channel, the wavelength, amplitude and phase of lee waves downstream are predicted with reasonable accuracy for mode 1 (with $\frac{h}{D} < 0.14$) by both the LBC and quasi-linear theories.

ii.   The LBC theory fails to predict the observed motions upstream; the agreement with the quasi-linear theory is fair, but so also is that with the quite different source-like model of Wong & Kao (1970), and this upstream comparison is therefore inconclusive. The upstream motions for mode 1 may become transient over the obstacle when K > 1.5, apparently because of non-linear effects over the obstacle associated with higher order modes.

iii.   For K near unity non-linear effects must be important, and the flow has some similarity with classical hydraulics.

6   ACKNOWLEDGEMENTS

7   APPENDIX - SYMBOLS

x   horizontal coordinate
z   vertical   "
t   time
u   horizontal velocity perturbation
w   vertical   "   "
ψ stream function   "
U   mean horizontal velocity of fluid relative to obstacle
N   Brunt-Väisälä frequency
D   channel depth
h   obstacle height
a   obstacle half-width
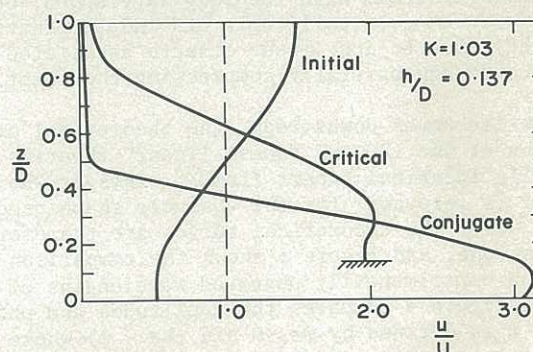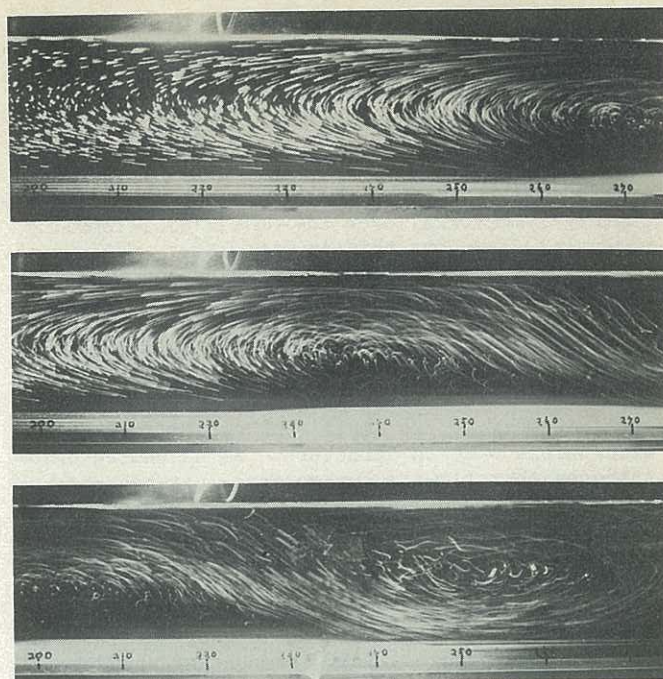J_1 Bessel function
K $= ND/\pi U$

Figure 6   Velocity profiles obtained from the hydrostatic model for a case with K near unity. These correspond approximately to positions indicated by the arrow in the three photographs.

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