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A New Approach to the Boundary Layer Equations : Optimal Control of the Von Mises Equation by Wall Suction and Numerical Solution Using Finite Difference and Finite Element Techniques

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SUMMARY The equations of an incompressible, steady bidimensional laminar boundary layer are dealt with in the form of Von Mises variables and are controlled by wall suction or blowing. The cost function is minimized by using the algorithm of steepest descent. The calculation of the gradient of the cost function is carried out by introducing an adjoint state and using a finite difference scheme. A finite element technique, to solve the same equations, is also outlined and a comparison between finite difference and finite element numerical solutions is presented for one pressure gradient flow with wall suction. Some preliminary results of the optimal control are also pointed out.

1 INTRODUCTION

During last years, the methods of optimal control (Lions, 1968) have had numerous applications such as the optimal control of non linear problems, of variational inequalities, control of variable domains, and control of eigenvalues. The field of applications has also been greatly extended to many branchs of mathematical physics as for example the physics of plasma, oceanography and fluid mechanics (Glowinski-Pironneau, 1975). In this paper, we shall study a fluid mechanics problem wich concerns the development of a boundary layer flow subjected to wall suction or blowing and we shall apply the optimal control technique to the governing equations after having recasted them into (Schlichting,1968), the Von Mises form. The control will be exerted by the suction or blowing at the wall to attain certain desired conditions of the velocity profile at a certain station.

In a first stage the development of the boundary layer will be controlled by the means of the rate of aspiration or blowing at the wall. In a second future stage, the control will be applied through the stream wise pressure gradient exerted by the outer flow. This is directly related to the outer velocity gradient at the edge of the boundary layer. In the following, the definitions, the formulation of the optimal control problem as well as related subjects such as the discretisation of the equations using finite differences will be explained. The results of calculation, without control, for different values of the pressure gradient parameter and wall suction is compared with a finite element technique which is given in the text. Some first results of the optimal control is shown also.

2 DEFINITION - MAIN EQUATIONS

The problem of optimal control applied to boundary layer development as stated above can be defined as follows. Consider a fixed station at $x = X$, where we want to get a "desired" velocity profile $U_d\ (X,y)$. An optimal distribution of the velocity of suction or blowing $v_o\ (x)$ is thus looked for to approach at the best this desired velocity profile U_d. It must be noted however that such a profile is not always realisable.

Let us consider the boundary layer equations (figure 1) "Prandtl equations". After introducing

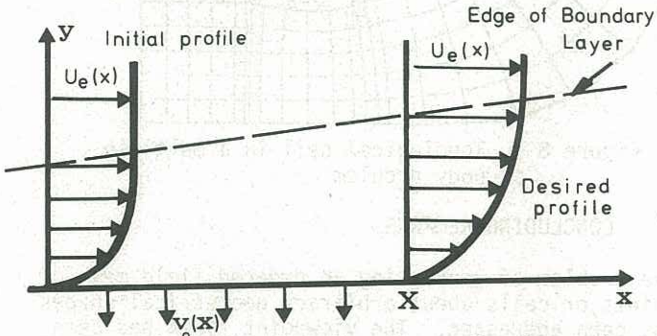

Figure 1 Boundary layer development

a modified Von Mises variables defined by :

$$u = \frac{\partial\psi}{\partial y}\ ;\ v - v_o(x) = -\frac{\partial\psi}{\partial x} \qquad (1)$$

the boundary layer equations can be put into the form :

$$\frac{\partial u}{\partial x} - \nu\frac{\partial}{\partial\psi}\left(u\frac{\partial u}{\partial\psi}\right) + v_o\frac{\partial u}{\partial\psi} + \frac{1}{u}\frac{dp}{dx} = 0 \qquad (2)$$

where u and v are the velocity components respectively in the stream wise direction x and its normal y ; p denotes the pressure and ν the kinematic viscosity. The density of the fluid is taken as unity. The boundary conditions associated with equation (2) are :

$$u(x,o) = 0\ ;\ v(x,o) = v_o(x)$$
$$u(x,\infty) = U_e(x) \qquad (3)$$

and the initial condition is given by :

$$u(o,\psi) = U_o(\psi) \qquad (4)$$

where $U_e(x)$ represents the local velocity outside the boundary layer (see figure 1) and $v_o(x)$ is the normal suction or blowing velocity at the wall. Bernoulli equation which is valid at the outer edge of the boundary layer is given by :

$$\frac{dp}{dx} + U_e \frac{dU_e}{dx} = 0 \qquad (5)$$

The advantage of the formulation (2) compared to the classical form of Von Mises equation (defined by the function $f = u^2/2$) lies in the operator $\partial/\partial\psi(u.\ \partial u/\partial\psi)$. It can be seen that the non homogeneous boundary condition at $\psi = \infty$ is automatically satisfied if $U_o(\infty) = U_e(o)$ and if $\partial u/\partial\psi$ and $\partial^2 u/\partial\psi^2$ are integrable in the neighbourhood of infinity.

3 FORMULATION OF THE OPTIMAL CONTROL PROBLEM

The equations (2), (3) and (4) are called the state equations of the system. The function $v_o = v_o(x)$ is the control and the set of admissible controls are :

$$\mathcal{U}_{ad} = \{v_o,\ 0 \leqslant v_o(x) \leqslant M\} \quad \text{(blowing)} \qquad (6)$$

or

$$\mathcal{U}_{ad} = \{v_o, -M \leqslant v_o(x) \leqslant 0\} \quad \text{(suction)} \qquad (7)$$

Let the position X>o be fixed and let the desired velocity profile $U_d = U_d(\psi)$ be given. One can introduce the cost function $J(v_o)$ defined as

$$J(v_o) = \int_o^\infty (u(v_o,X) - U_d)^2\ d\psi \qquad (8)$$

which will be minimised on the set $\mathcal{U}_{ad}$. The control function $\hat{V}_o$ which realises this minimum is called the optimal control. Note that unicity of the solution is not guaranted.

The real problem is to characterize an optimal control. Unfortunately, one has only necessary conditions of optimality based on the calculation of the gradient of the function $J(v_o)$. Thus we shall give here a simple mean to calculate the gradient $J'(v_o)$. For this, we introduce an adjoint state $q = q(v_o)$ defined as the solution of the equation

$$\frac{\partial q}{\partial x} + \nu u(v_o) \frac{\partial^2 q}{\partial \psi^2} + v_o \frac{\partial q}{\partial \psi} + \frac{q}{u^2(v_o)} \frac{dp}{dx} = o \qquad (9)$$

with the conditions

$$q(x,o) = q(x,\infty) = o \qquad (10)$$

$$q(X,\psi) = u(v_o,X) - U_d \qquad (11)$$

Thus the gradient J' can be written as

$$J'(v_o) = -2\int_o^\infty q(v_o) \frac{\partial u}{\partial \psi}(v_o)\ d\psi \qquad (12)$$

It can be noted that q is the solution of a reversed evolution problem.

4 NUMERICAL ALGORITHM

One is looking for an optimal control with the aid of a descent method of the type "projected gradient". This is a simple and a rapid method of calculation, and it seems unnecessary to utilize a more sophisticated method because the cost function is not a convex one.

The principle of the employed algorithm is as follows. Let an approximation v_o^k be calculated where k denotes the level of iteration at which calculation is carried out ; thus one can resolve the state equations (2), (3), (4) which yield the solution u^{k+1}. The adjoint state equations (9), (10), (11) can in turn be solved to give q^{k+1}. Using equation (12), the gradient $J'(v_o^k)$ can be calculated. Writing

$$w_o^{k+1} = v_o^k - \eta\ J'(v_o^k) \qquad (13)$$

A major difficulty is the determination of the optimal η as indicated by CEA (1971).

Finally, the wall suction or blowing velocity v_o^{k+1} at the level k+1 is calculated as the projection on the set of admissible control of w_o^{k+1}

$$v_o^{k+1} = \text{Proj.}_{/\mathcal{U}_{ad}}\ w_o^{k+1} \qquad (14)$$

In practice, for the determination of the adjoint state, it is preferable to calculate the adjoint of the discretisation of equations (2), (3), (4) instead of discretisating equations (9), (10), (11)

5 DISCRETISATION OF THE STATE EQUATION AND OF THE ADJOINT EQUATION

If the indices i and n are affected respectively to the variables ψ and x, the state equation (2) can be discretisated using finite differences. This leads to the following equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta x} = \frac{\nu}{2\Delta\psi^2}\left[\theta\left[\delta^2(u^2)_i^{n+1}\right] + (1-\theta)\left[\delta^2(u^2)_i^n\right]\right] - v_o^{n+1}\frac{(\delta u)_i^{n+1}}{\Delta\psi} - \frac{1}{u_i^j}\left(\frac{dp}{dx}\right)^{n+1} \qquad (15)$$

where the parameter θ is in the interval (0,1) ; i can take the values 1,2,...(I-1), whereas n varies from 0 to (N-1), where I and N are the number of grid points taken for ψ and x respectively. For the case of constant step of ψ variable ($\Delta\psi$ = constant), case that we shall admit to simplify the equations, the first and second differences take the form :

$$(\delta u)_i^{n+1} \simeq u_{i+1}^{n+1} - u_{i-1}^{n+1}$$
$$(\delta^2 u)_i^{n+1} \simeq u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} \qquad (16)$$

The term $(u^2)_i^{n+1}$ is linearized in the following manner :

$$(u^2)_i^{n+1} \simeq 2\ u_i^n\ u_i^{n+1} - (u_i^n)^2 \qquad (17)$$

The system (15) can thus be written on the form :

$$a_i\ u_{i+1}^{n+1} + b_i\ u_i^{n+1} + c_i\ u_{i-1}^{n+1} = d_i \qquad (18)$$

where

$$a_i = \frac{v_o^{n+1}}{2\Delta\psi} - \theta\nu\frac{u_{i+1}^n}{\Delta\psi^2}$$

$$b_i = \frac{1}{\Delta x} + 2\ \theta\nu\frac{u_i^n}{\Delta\psi^2}$$

$$c_i = \frac{v_o^{n+1}}{2\ \Delta\psi} - \nu\theta\frac{u_{i-1}^n}{\Delta\psi^2}$$

$$d_i = \frac{u_i^n}{\Delta x} + \nu\frac{1-2\theta}{\Delta\psi^2}\left((u_{i+1}^n)^2 - 2\ (u_i^n)^2 + (u_{i-1}^n)^2\right) - \frac{1}{u_i^n}\left(\frac{dp}{dx}\right)^{n+1}$$

and i = 1, ..., I-1 ; n = 1,..., N-1

and $a_1 = C_{I-1} = 0$ by virtue of the boundary conditions defined alongwith initial conditions by equations (3) and (4).

The variational formulation associated with equation (15) can be obtained by multiplying equation (15) by q_i^{n+1}. $\Delta x.\Delta\psi$ and by taking the sum on i and n. Here, q_i^{n+1} represents the discretisation of the adjoint state q introduced into equation (9). Thus we get for all q_i^{n+1} the following equation :

$$\sum_{n=o}^{N-1}\sum_{i=1}^{I-1}\Big[(u_i^{n+1}-u_i^n)q_i^{n+1}\,\Delta\psi - \theta\nu\frac{\Delta x}{\Delta\psi}q_i^{n+1}\,(u_{i+1}^{n+1}u_{i+1}^n - 2u_i^{n+1}u_i^n + u_{i-1}^{n+1}u_{i-1}^n) - \frac{\nu(1-2\theta)\Delta x}{2\Delta\psi}q_i^{n+1}\big[(u_{i+1}^n)^2-2(u_i^n)^2+(u_{i-1}^n)^2\big] + v_o^{n+1}\frac{\Delta x}{2}q_i^{n+1}(u_{i+1}^{n+1}-u_{i-1}^{n+1}) + \frac{1}{u_i^n}q_i^{n+1}\left(\frac{dp}{dx}\right)^{n+1}\Delta x\Delta\psi\Big] = o \quad (19)$$

Now, one can give an elementary increase δv_o, for which corresponds the elementary increase of the velocity δu. Neglecting terms of second order, the vector $(\delta u)_i^n$ is obtained as the solution of the next equation (Brauner, Gay, 1977).

$$\sum_{n=o}^{N-1}\sum_{i=1}^{I-1}\Big\{(\delta u_i^{n+1}-\delta u_i^n)q_i^{n+1}\,\Delta\psi - \theta\nu\frac{\Delta x}{\Delta\psi}q_i^{n+1}(\delta u_{i+1}^{n+1}.u_{i+1}^n + u_{i+1}^{n+1}.\delta u_{i+1}^n - 2\,\delta u_i^{n+1}.u_i^n - 2u_i^{n+1}.\delta u_i^n + u_{i-1}^n.\delta u_{i-1}^{n+1} + u_{i-1}^{n+1}.\delta u_{i-1}^n) - \nu(1-2\theta)\frac{\Delta x}{\Delta\psi}q_i^{n+1}(u_{i+1}^n.\delta u_{i+1}^n - 2\,u_i^n.\delta u_i^n + u_{i-1}^n.\delta u_{i-1}^n) + v_o^{n+1}\frac{\Delta x}{2}q_i^{n+1}(\delta u_{i+1}^{n+1}-\delta u_{i-1}^{n+1}) - \frac{1}{(u_i^n)^2}\delta u_i^n\,\Delta x\,q_i^{n+1}\left(\frac{dp}{dx}\right)^{n+1}\Big\}$$

$$= -\sum_{n=o}^{N-1}\sum_{i=1}^{I-1}\delta v_o^{n+1}\frac{\Delta x}{2}q_i^{n+1}(u_{i+1}^{n+1}-u_{i-1}^{n+1}) \quad (20)$$

with the conditions on δu

$$\delta u_o^{n+1} = \delta u_I^{n+1} = \delta u_i^o = 0$$

The adjoint state q_i^{n+1} is chosen in a manner that the left and side of equation (20) is equal to the increase δJ of the cost function. If the cost function $J(v_o)$ is approximated by

$$J(v_o) = \sum_{i=1}^{I-1}(u_i^N - u_{id})^2\,\Delta\psi \quad (21)$$

the increase δJ is thus given by

$$\delta J = 2\sum_{i=1}^{I-1}(u_i^N - u_{id})\,\delta u_i^N\,\Delta\psi \quad (22)$$

also we have :

$$\delta J = \sum_{n=o}^{N-1}\frac{\partial J}{\partial v^n}\,\delta v_o^n$$

thus by identification with the right hand side of equation (20), we obtain :

$$\frac{\partial J}{\partial v^n} = -\sum_{i=1}^{I-1}\Delta x\,q_i^n(u_{i+1}^n - u_{i-1}^n) \quad (23)$$

$$\frac{\partial J}{\partial v_o^o} = o$$

After rearrangement, the reverse equation statisfied by the adjoint state q can be put on the form:

$$\alpha_n\,q_{i+1}^n + \beta_n\,q_i^n + \gamma_n\,q_{i-1}^n = \lambda_n \quad (24)$$

where
$$\alpha_n = -\nu\theta\frac{u_i^{n-1}}{\Delta\psi^2} - \frac{v_o^n}{2\Delta\psi}$$

$$\beta_n = \frac{1}{\Delta x} + 2\nu\theta\frac{u_i^{n-1}}{\Delta\psi^2}$$

$$\gamma_n = -\nu\theta\frac{u_i^{n-1}}{\Delta\psi^2} + \frac{v_o^n}{2\Delta\psi}$$

$$\lambda_n = q_{i+1}^{n+1}\left(\nu\theta\frac{u_i^{n+1}}{\Delta\psi^2} + (1-2\theta)\,\nu\frac{u_i^n}{\Delta\psi^2}\right) + q_i^{n+1}\left(\frac{1}{\Delta x} - 2\nu\frac{\theta u_i^{n+1}+(1-2\theta)u_i^n}{\Delta\psi^2} + \frac{1}{(u_i^n)^2}\left(\frac{dp}{dx}\right)^{n+1}\right)$$

$$\lambda_N = 2\,\frac{u_i^N - u_{id}}{\Delta x}$$

and $i = 1,\ldots, I-1$; $n = 1,\ldots, N$.

The boundary conditions are $q_o^n = q_I^n = 0$ for all n.

It is to be noted that equation (24) is nothing else than the discretized form of equation (9). The systems of equations (18), (24) being linear and tridiagonal are solved using the method of Gauss.

6 FINITE ELEMENT SOLUTION OF VON MISES EQUATION

In order to get a solution for the Von Mises equation we have proceeded by two methods : the finite difference one explained above, and the finite element one ; the details of which are given below (Gay, 1977 ; Assassa, 1977). No optimal control is introduced into the equation, at this stage, as was the case for the finite difference solution.

Defining the function $f(x,\psi) = u^2/2$, equation (2) takes the form :

$$\frac{\partial f}{\partial x} + v_o(x)\,\frac{\partial f}{\partial\psi} = \frac{1}{2}\frac{d}{dx}(U_e^2) + \nu\,\sqrt{2f}\,\frac{\partial^2 f}{\partial\psi^2} \quad (25)$$

The boundary conditions are thus given by :

$\psi = 0$; $f = 0$

$\psi = \delta(x)$; $f = U_e^2/2$

where, for the numerical calculation, the boundary condition at infinity is replaced by the one taken at the edge of the boundary layer.

In the present work, the finite element approach is based on the method of Galerkin (Bonnerot, Jamet, 1975). Thus using certain shape functions $\Phi(x,\psi)$ which vanish at the wall and at the edge of the boundary layer $\delta(x)$, equation (25) after multiplication by $\Phi(x,\psi)$, integration on the element $[(x_1,x_2), (0,\delta(x))]$ and integration by parts can be written in the following form :

$$-\int_{x_1}^{x_2}\int_0^{\delta(x)} f\,\frac{\partial\Phi}{\partial x}\,d\psi dx + \int_0^{\delta(x_2)}\Phi(x_2,\psi)f(x_2,\psi)d\psi - \int_0^{\delta(x_1)}\Phi(x_1,\psi)f(x_1,\psi)d\psi - \int_{x_1}^{x_2}\int_0^{\delta(x)} f\,v_o(x)\,\frac{\partial\Phi}{\partial\psi}\,d\psi dx - \frac{1}{2}\int_{x_1}^{x_2}\int_0^{\delta(x)}\Phi\,\frac{d}{dx}U_e^2\,d\psi dx$$

$$= -\int_{x_1}^{x_2}\int_0^{\delta(x)} \frac{\partial f}{\partial \Psi}\frac{\partial \Phi}{\partial \Psi}\sqrt{2f}\, d\Psi dx$$

$$- \int_{x_1}^{x_2}\int_0^{\delta} \Phi\, \frac{(\partial f/\partial \Psi)^2}{\sqrt{2f}}\, d\Psi dx \qquad (26)$$

The different integrals of the equation (26) are calculated numerically using the transformation shown in figure (2)

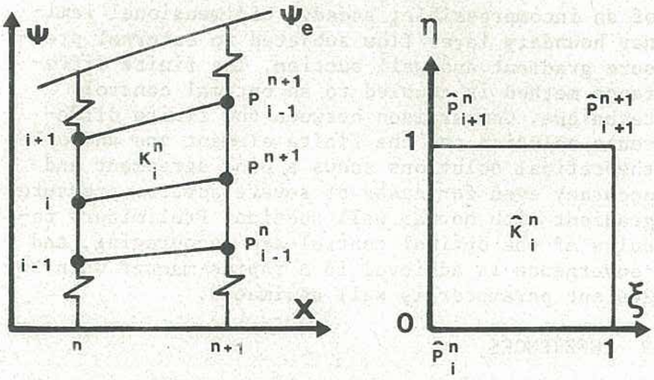

Figure 2 Transformation of the element K_i^n

An element K_i^n in the plane (x,Ψ) is tranformed to a square of unit length in the plane (ξ,η) using the transformation :

$$x = (1-\xi)x^n + \xi\, x^{n+1} \qquad (27.a)$$

$$\Psi = (1-\eta)\left[(1-\xi)\Psi_i^n + \xi\, \Psi_i^{n+1}\right] + \eta\left[(1-\xi)\Psi_{i+1}^n + \xi\Psi_{i+1}^{n+1}\right] \qquad (27.b)$$

The Jacobian of this transformation is given by

$$d\Psi dx = J_i^n(\xi,\eta)\, d\xi d\eta = k(\Psi_{i+1}^{n+\xi} - \Psi_i^{n+\xi})$$

where k is the step in the x direction $k := x^{n+1} - x^n$ and

$$\Psi_i^{n+\xi} = (1-\xi)\Psi_i^n + \xi\, \Psi_i^{n+1}$$

In the plan (ξ,η) a function $F(\xi,\eta)$ defined in K_i^n will be approximated by the polynomial form :

$$F(\xi,\eta) = a + b\xi + c\eta + d\xi\eta \qquad (28)$$

It follows that F is determined by its four values at the corners of the square K_i^n : $[(0 \leqslant \xi \leqslant 1), (0 \leqslant \eta \leqslant 1)]$ as :

$$F(\xi,\eta) = (1-\eta)\,(1-\xi)\quad F(0,0) + (1-\eta)\xi F(0,1)$$
$$+ \eta(1-\xi)F(1,0) + \xi\eta F(1,1) \qquad (29)$$

As far as the integrals (26) in the system (ξ,η) are concerned, they are numerically calculated for each element K_i^n by the formula :

$$\iint_{K_i^n} F(x,\Psi)\, d\Psi dx = \iint_{\hat{K}_i^n} F(\xi,\eta) J_i^n d\xi d\eta$$
$$\simeq \frac{1}{4} \sum_{\substack{\xi=0,1\\ \eta=0,1}} F(\xi,\eta) J_i^n(\xi,\eta) \qquad (30)$$

where $\hat{K}_i^n$ is the element in the plane (ξ,η) that corresponds to the element K_i^n in the plane (x,Ψ). The shape functions Φ (ξ,η) used in the calculation are presented in figure 3.

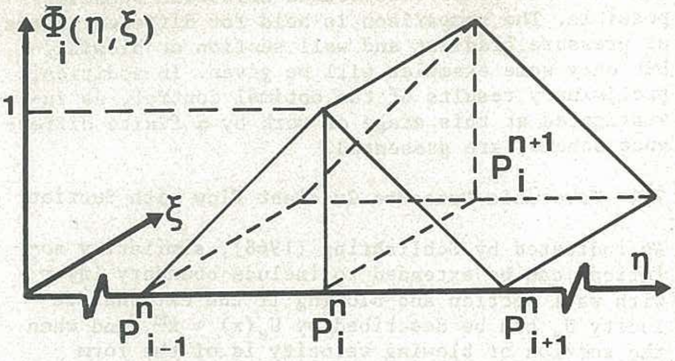

Figure 3 Shape function at the point P_i^n

Thus, the initial equation (25) can be put in the linear tradiagonal matrix form :

$$A_i\, f_{i-1}^{n+1} + B_i\, f_i^{n+1} + C_i\, f_{i+1}^{n+1} = D_i \qquad (31)$$

with i = 2, 3,..., (I-1) where I is the number of nodes taken accross the boundary layer. The coefficients of equation (31) are given by :

$$A_i = \frac{1}{4}(\Psi_{i-1}^{n+1} - \Psi_i^n) - \frac{k}{4} v_o^{n+1} - \frac{\nu k}{4}\frac{s_{i-1}^n + s_i^n}{\Psi_i^{n+1} - \Psi_{i-1}^{n+1}} - \frac{\nu k}{4 s_i^n}\frac{f_i^n - f_{i-1}^n}{\Psi_i^n - \Psi_{i-1}^n}$$

$$B_i = \frac{\nu k}{4}\left[\frac{s_i^n + s_{i+1}^n}{\Psi_{i+1}^{n+1} - \Psi_i^{n+1}} + \frac{s_{i-1}^n + s_i^n}{\Psi_i^{n+1} - \Psi_{i-1}^{n+1}}\right] + \frac{\nu k}{4 s_i^n}\left[\frac{f_i^n - f_{i-1}^n}{\Psi_i^n - \Psi_{i-1}^n} - \frac{f_{i+1}^n - f_i^n}{\Psi_{i+1}^n - \Psi_i^n}\right]$$
$$+ \frac{1}{2}(\Psi_{i+1}^{n+1} - \Psi_{i-1}^{n+1})$$

$$C_i = \frac{-1}{4}(\Psi_{i+1}^{n+1} - \Psi_{i+1}^n) + \frac{k}{4} v_o^{n+1} - \frac{\nu k}{4}\frac{s_{i-1}^n + s_i^n}{\Psi_i^{n+1} - \Psi_{i-1}^{n+1}}$$
$$- \frac{\nu k}{4 s_i^n}\frac{f_i^n - f_{i-1}^n}{\Psi_i^n - \Psi_{i-1}^n}$$

$$-D_i = \frac{-1}{4} f_{i+1}^n (\Psi_{i+1}^{n+1} - \Psi_{i+1}^n) + \frac{1}{4} f_{i-1}^n (\Psi_{i-1}^{n+1} - \Psi_{i-1}^n)$$
$$+ \frac{k}{4} v_o^n (f_{i+1}^n - f_{i-1}^n) + \frac{\nu k}{4}\left[(s_{i-1}^n + s_i^n)\frac{f_i^n - f_{i-1}^n}{\Psi_i^n - \Psi_{i-1}^n}\right.$$
$$\left. - (s_i^n + s_{i+1}^n)\frac{f_{i+1}^n - f_i^n}{\Psi_{i+1}^n - \Psi_i^n}\right] + \frac{\nu k}{4 s_i^n}\left[\frac{(f_{i+1}^n - f_i^n)^2}{\Psi_{i+1}^n - \Psi_i^n} + \frac{(f_i^n - f_{i-1}^n)^2}{\Psi_i^n - \Psi_{i-1}^n}\right]$$
$$- \frac{1}{2} f_i^n (\Psi_{i+1}^n - \Psi_{i-1}^n) + \frac{g^n - g^{n+1}}{8}\left[(\Psi_{i+1}^{n+1} - \Psi_{i-1}^{n+1})\right.$$
$$\left. + (\Psi_{i+1}^n - \Psi_{i-1}^n)\right]$$

with $g^n = (U_e^2/2)^n$ and $s_i^n = (\sqrt{2f})_i^n$. The last coefficient is modified to take into consideration the outer boundary condition.

7 SOME NUMERICAL RESULTS

Numerical computation of the finite difference and finite element solutions were carried out on C.D.C. 7600 computer using Fortran IV language programs. Numerical results of the finite difference solution are compared with those of the finite element solution and with other theoretical solutions whenever possible. The comparison is held for different cases of pressure gradient and wall suction or blowing, but only some examples will be given. In addition, preliminary results of the optimal control, as investigated at this stage of work by a finite difference scheme, are presented.

7.1 Favorable Pressure Gradient Flow with Suction

As indicated by Schlichting (1968), similarity solutions can be extended to include boundary layer with wall suction and blowing if the external velocity U_e can be described by $U_e(x) \sim x^m$, and when the suction or blowing velocity is of the form $v_o(x) \sim x^{0.5(m-1)}$. Calculation was carried out with $v_o(x) = - C\sqrt{\nu U_e}x^{0.5(m-1)}$ for a suction case with C = 1. The input initial velocity profile was taken as Blasius profile (zero pressure gradient without suction) and this profile was strained by the double effect of the imposed pressure gradient, β = 0.8 (β is the Falkner-Skan parameter) and wall suction leading to a similarity solution as presented in figure 4 for the finite element and finite difference solutions.

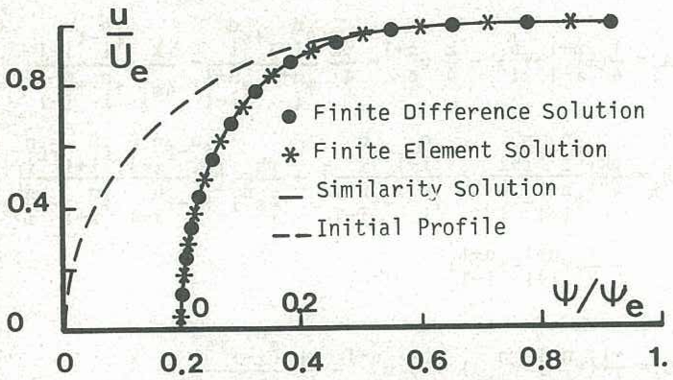

Figure 4 Comparaison betwen finite difference and finite element solutions with similarity one

7.2 Optimal Control Performance

This is a test case for the algorithm of optimal control used in this work. The desired velocity profile is the one given in figure 4. Initial suction velocity was taken to be zero. The iteration procedure is illustrated on figure 5, where is displayed three suction distributions corresponding to three different iterations. The numerical values of the cast function $J(v_o)$ are is follows : for the initial zero suction distribution of iteration level zero, and for a number of iterations equals to 10 then 25, J varies respectively as 31×10^{-6}, 2.82×10^{-6}, and 2.02×10^{-6}. This shows the rapid variation of J towards a minimum during the first iterations.

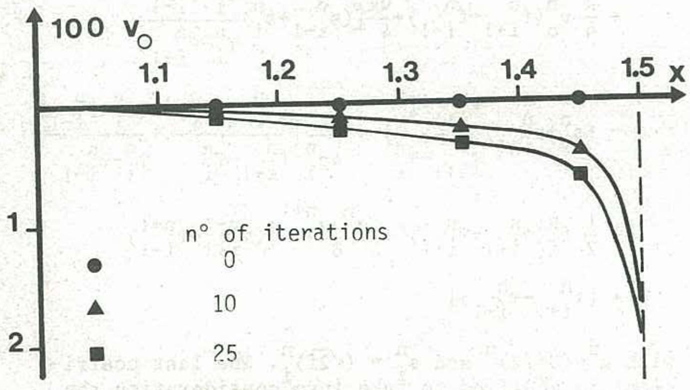

Figure 5 Suction Control

8 CONCLUSION

A finite element method as well as a finite difference one were developed to describe the behaviour of an incompressible, steady, bidimensional laminar boundary layer flow subjeted to external pressure gradient and wall suction. The finite difference method is coupled to an optimal control technique. Comparaison between the finite difference solution and the finite element one and other theoretical solutions shows a good agreement and accuracy even for cases of severe adverse pressure gradient with normal wall suction. Preliminary results of the optimal control are encouraging, and convergence is achieved in a rapide manner when the descent parameter is well estimated.

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