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Flow Towards a Well of Large Diameter in a Leaky Confined Aquifer

M. H. ABDUL KHADER
Senior Lecturer, Department of Civil Engineering, University of Singapore, Singapore
and
D. RAMADURGAIAH
Assistant Professor, Department of Civil Engineering, College of Engineering, Anantapur, India

SUMMARY An exact solution is presented to compute the unsteady drawdown in and around a fully penetrating, steadily discharging well in a leaky confined aquifer. The aquifer is assumed to be homogeneous, isotropic and infinite in lateral extent. The effect of storage capacity of the well on the unsteady flow pattern is considered. The Laplace transformation and the complex inversion theorem are used in the mathematical development. The drawdown solution is numerically evaluated using a high-speed computer and the results are presented in graphs.

1 INTRODUCTION

The present practice of aquifer evaluation is based essentially on the application of two unsteady drawdown equations, viz: the Theis formula [6] for a non-leaky aquifer and the Hantush-Jacob formula [2] for a leaky aquifer. Both these formulae have been developed by treating the well as a mathematical line-sink. Papadopulos and Cooper [5] have, by introducing a factor to define the rate of depletion of water stored in the well, shown that the line-sink formulae are inadequate to predict the drawdown at early periods of pumping and for large well diameters. The Papadopulos-Cooper equation is valid for a non-leaky aquifer.

The purpose of this paper is to present a transient drawdown formula of a general nature for the problem of a completely penetrating artesian well of constant discharge. The well is considered to be of finite diameter. Factors defining the loss of storage of the well and leakage through the aquitard are included in formulating the problem. The aquifer is assumed to be isotropic and to extend laterally to infinity. Water can enter the well only from the confined aquifer. Water can enter this aquifer through an aquitard. The motion of groundwater is assumed to be governed by Jacob's model of linear leakage [3]. Well losses are ignored.

2 THEORETICAL DEVELOPMENT

2.1 Statement of the Problem

The system of radial flow towards a well fully penetrating a leaky confined aquifer is shown in Figure 1. The flow around the well is governed by the following equations:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{B^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}, \quad r \geq r_w \qquad (1)$$

$$s(r,0) = 0 \qquad (2)$$

$$s(\infty,t) = 0 \qquad (3)$$

and

$$2\pi r_w \, Kb \frac{\partial}{\partial r} s(r_w,t) - \pi r_w^{\,2} \frac{\partial}{\partial t} s(r_w,t) = - Q, \quad t \geq 0 \qquad (4)$$

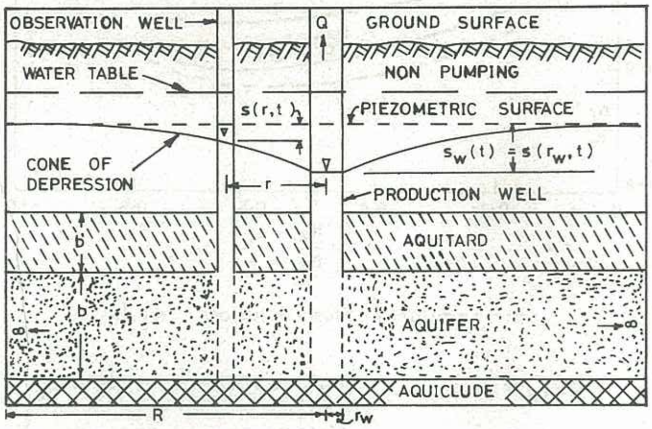

Figure 1 Definition sketch of a fully penetrating well in a leaky confined aquifer

in which $s(r,t)$ = drawdown at radial distance r from the axis of the well at time t after the commencement of pumping; $B=(Kbb'/K')^{\frac{1}{2}}$ = a factor of leakage; $\nu=Kb/S$ = hydraulic diffusivity of aquifer; b,b' = thicknesses of aquifer and aquitard respectively; K,K' = hydraulic conductivities of aquifer and respectively; S = storage coefficient of aquifer; r_w = radius of well and Q = constant rate at which the well is pumped.

2.2 Solution to the Problem

Application of the Laplace transformation with respect to time to the problem reduces the partial differential equation (1) to an ordinary differential equation which is then solved to satisfy the transformed boundary conditions. The solution thus obtained, when inverted for the Laplace transformation, gives the required drawdown function.

Applying the Laplace transformation with respect to t to Eq.(1) and making use of Eq.(2) yields the following modified Bessel equation:

$$\frac{d^2\bar{s}}{dr^2} + \frac{1}{r}\frac{d\bar{s}}{dr} - q^2\bar{s} = 0, r \geq r_w \qquad (5)$$

in which

$$q = \left(\frac{p}{\nu} + \frac{1}{B^2}\right)^{\frac{1}{2}}$$

and

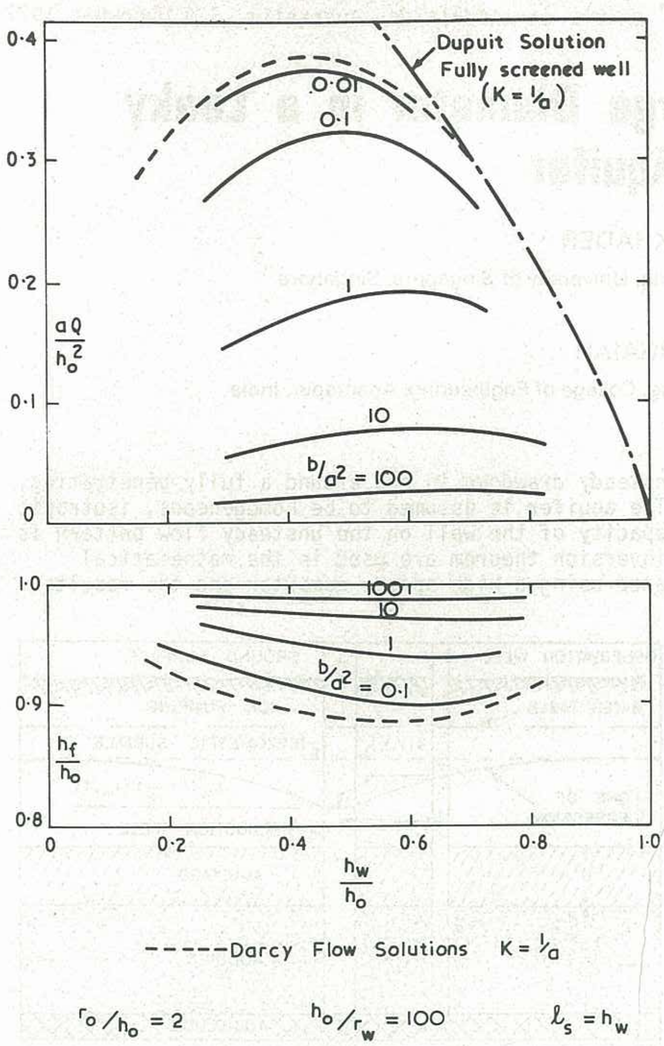

Figure 3 Typical well performance curves

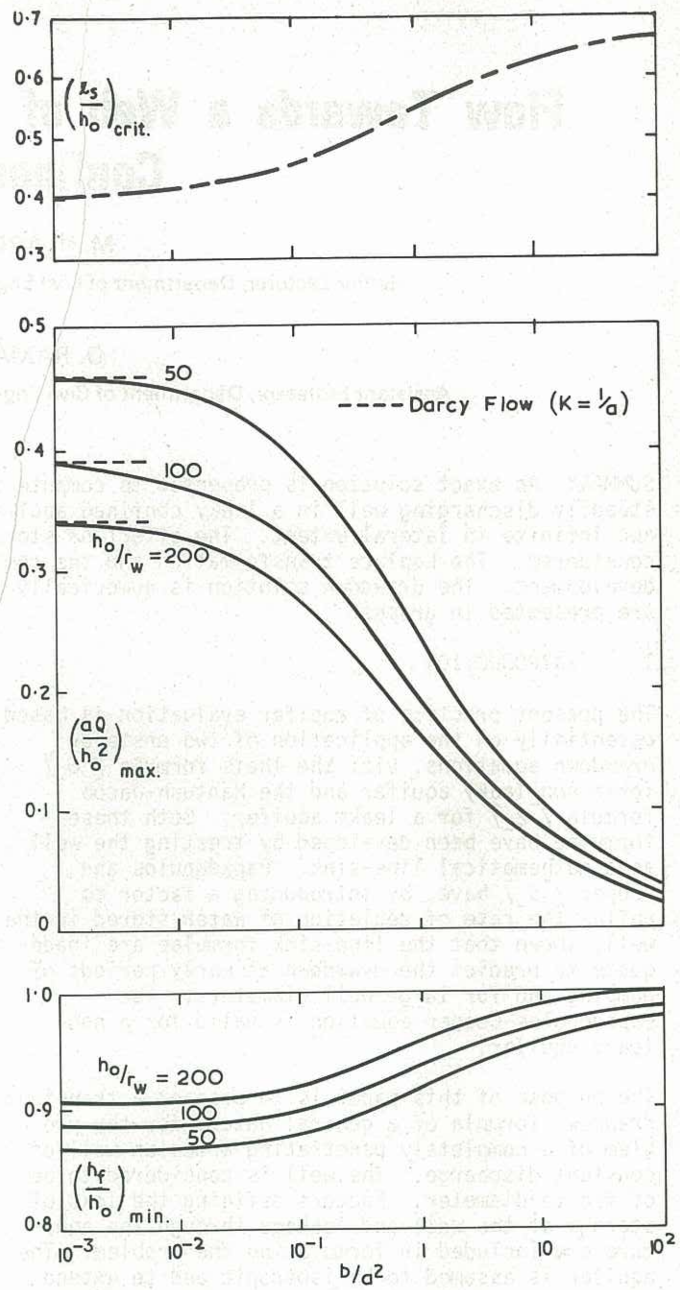

Figure 4 Maximum discharge and drawdown

7 CONCLUSION

In the past, lack of data has prevented engineers and hydrogeologists engaged in well design from making comprehensive examinations of the effects of well and aquifer variables on well performance in other than simple cases. It is considered that the results of this investigation will be useful in selecting optimal designs for partially screened wells in unconfined aquifers.

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$$\bar{s}(r,p) = \int_0^\infty \exp(-pt)\, s(r,t)\, dt$$

Eqs. 3 and 4 are similarly transformed to

$$\bar{s}(\infty,p) = 0 \tag{6}$$

and

$$2\pi r_w\, Kb \frac{d}{dr}\bar{s}(r_w,p) - \pi r_w^2\, p\bar{s}(r_w,p) = -\frac{Q}{p} \tag{7}$$

The general solution to Eq.(5) is

$$\bar{s} = C_1 I_0(qr) + C_2 K_0(qr) \tag{8}$$

in which I_0 and K_0 are the zero order modified Bessel functions of the first and second kinds respectively. Using Eqs. (6) and (7) and noting that $I_0(\infty) = \infty$, $K_0(\infty) = 0$ and $d\{K_0(qr)\}/dr = -qK_1(qr)$, the values of C_1 and C_2 are determined. Thus,

$$\bar{s} = \frac{Q\, K_0(qr)}{p\{2\pi r_w\, Kb\, qK\,(qr_w) + \pi r_w^2 p\, K_0(qr_w)\}} \tag{9}$$

in which K_1 = first order modified Bessel function of the second kind.

The drawdown distribution $s(r,t)$ in the aquifer is the inverse Laplace transform of Eq. (9). Application of the complex inversion theorem [1] to Eq. (9) gives

$$s = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} Q\, K_0(\mu r)\exp(\lambda t)/$$

$$\lambda\{2\pi Kb\mu r_w\, K_1(\mu r_w) + \pi r_w^2\, \lambda K_0(\mu r_w)\}.d\lambda \tag{10}$$

which, after rearranging the terms may be written in the following form:

$$\frac{s}{(Q/4\pi Kb)} = \frac{2S}{\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_0(\mu r)\exp(\lambda t)/$$

$$2S\lambda\mu r_w\, K_1(\mu r_w) + \frac{r_w^2}{\nu}\lambda^2\, K_0(\mu r_w).d\lambda \tag{11}$$

where $\mu = (\frac{\lambda}{\nu} + \frac{1}{B^2})^{\frac{1}{2}}$

The integrand of Eq. (11) has a pole at $\lambda = 0$ and a branch point at $\lambda = -\nu/B^2$. The contour given in Figure 2 (with $\lambda_b = \nu/B^2$) is used to evaluate the integral. It can be readily shown that in the limit when the radius of the large circle M tends to ∞ and that of the small circle N tends to zero, the sum of the line integral of Eq. (11) and the integrals along the branch lines EF and GH must equal the sum of the residues at the poles of the integrand of Eq. (11). The residue of this integrand at its only pole: $\lambda = 0$ is

$$\frac{2\, K_0(r/B)}{(r_w/B)\, K_1(r_w/B)} \tag{12}$$

The integrals along EF and GH are evaluated by taking

$$\lambda = \nu\{\frac{h^2}{r_w^2} + \frac{1}{B^2}\}\, e^{i\pi} \text{ on EF}$$

and $$\lambda = \nu\{\frac{h^2}{r_w^2} + \frac{1}{B^2}\}\, e^{-i\pi} \text{ on GH,}$$

and by using the identities [4]:

$$K_0(h\, e^{i\pi/2}) = -\frac{i\pi}{2}\{J_0(h) - i\, Y_0(h)\},$$

$$K_0(h\, e^{-i\pi/2}) = \frac{i\pi}{2}\{J_0(h) + i\, Y_0(h)\},$$

$$K_1(h\, e^{i\pi/2}) = -\frac{\pi}{2}\{J_1(h) - i\, Y_1(h)\} \text{ and}$$

$$K_1(h\, e^{-i\pi/2}) = -\frac{\pi}{2}\{J_1(h) + i\, Y_1(h)\} \tag{13}$$

in which h is a real positive number and J_n and Y_n are n-th order Bessel functions of kinds one and two respectively. Thus, after introducing the following non-dimensional ratios:

$$\delta = r_w/B, \quad \rho = r/r_w \text{ and } \tau = \nu t/r_w^2,$$

the solution is

$$\frac{s}{(Q/4\pi Kb)} = \frac{2K_0(\rho\delta)}{\delta K_1(\delta)} - \frac{8s}{\pi}\int_0^\infty$$

$$\frac{\exp\{-\tau(\delta^2+h^2)\}\, h\, U(h)}{(\delta^2+h^2)\, V(h)}\, dh \tag{14}$$

in which

$$U = J_0(\rho h)\{(\delta^2+h^2)\, Y_0(h) - 2Sh\, Y_1(h)\}$$

$$- Y_0(\rho h)\{(\delta^2+h^2)\, J_0(h) - 2Sh\, J_1(h)\}$$

and

$$V = \{(\delta^2+h^2)\, J_0(h) - 2Sh\, J_1(h)\}^2$$

$$+ \{(\delta^2+h^2)\, Y_0(h) - 2Sh\, Y_1(h)\}^2$$

Eq. (14) is the general equation for drawdown distribution in an infinite leaky aquifer due to a fully penetrating well. The unsteady drawdown: $s_w(t) = s(r_w,t)$ in the well is obtained by substituting $\rho = 1$ in this equation. If the well has a non-uniform section with radii r_w and r_c, respectively within the aquifer and above the top of the aquifer, the drawdown may be obtained by multiplying the factor S in Eq. (14) by $(r_w/r_c)^2$.

For the case of zero leakage ($B=\infty$ or $\delta=0$), the preceding technique of contour integration is unsuitable as it will yield a loop integral $\{\infty, 0(+)\}$. It is necessary to use the following theorem of Laplace transformation [1]:

$$\int_0^\infty \exp(-pt)dt \int_0^t s(t')dt' = \bar{s}/p \tag{15}$$

in order to invert Eq. (9) (with $B=\infty$, i.e. $q^2 = p/\nu$) and obtain the drawdown expression in the form given by Papadopulos and Cooper [5].

3 NUMERICAL COMPUTATIONS

3.1 To enable practical application of the solution, Eq. (14) is programmed for the IBM 370/155 digital computer system at the Indian Institute of Technology, Madras. Results of numerical evaluation are presented in Figures 3 to 6.

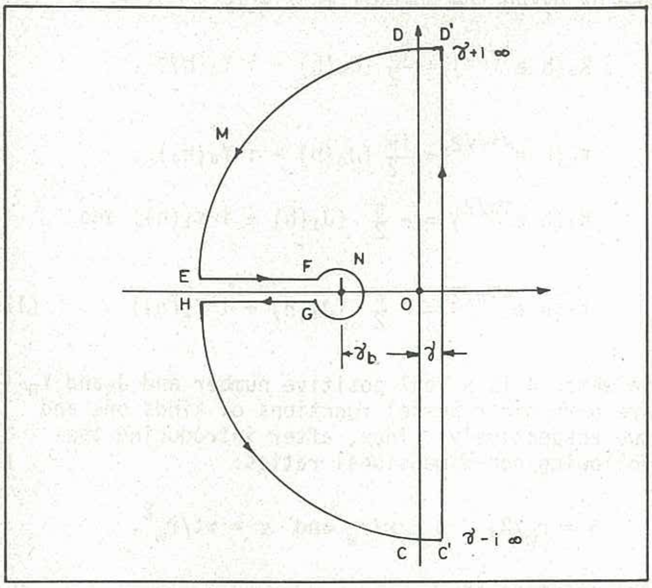

Figure 2 Contour for inversion of Laplace transform with a branch point on the negative real axis

Figure 3 provides a comparison between the present solution and the previously available transient drawdown formulae for a fully penetrating well. For computations of water level in the well, the non-leaky aquifer, large-well solution of Papadopulos-Cooper (curve BCE) is found to be useful for early periods of pumping, while the leaky aquifer, line-sink equation of Hantush-Jacob (curve ACD) is suitable for relatively large values of time. The present equation (curve BCD) in which the effects of both well storage and aquitard leakage are incorporated, is seen to offer the best results over the entire range of time.

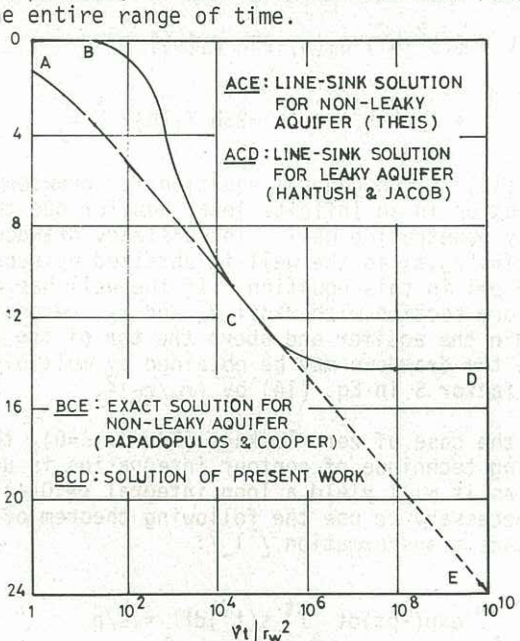

Figure 3 Time - drawdown variation in the well for $S=\delta=10^{-3}$

Graphs showing variation of water level in the well with time for various values of leakage ratio δ and storage coefficient S are given in Figures 4 and 5. Depending on the relative magnitude of the two factors, δ and S, drawdown in the well attains a steady state at a particular value of time. This is the stage beyond which the entire quantity of water pumped from the well is derived from leakage. Type curves on log-log scale can be readily constructed from these figures for analysis of data from field pumping tests. For the case of an infinite non-leaky aquifer, the radial distribution of drawdown around the well at different times is shown in Figure 6.

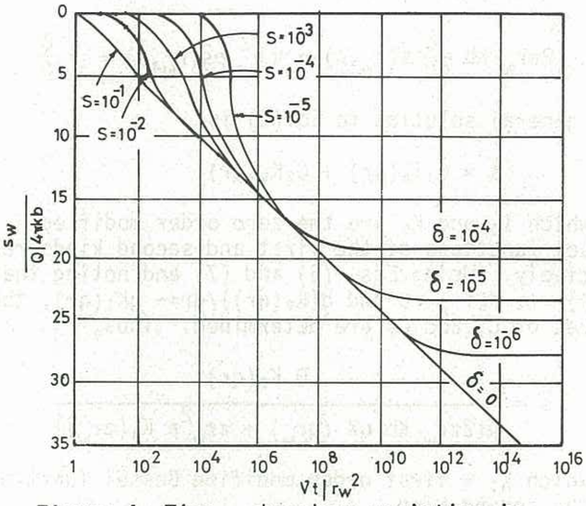

Figure 4 Time - drawdown variation in the well for $\delta < 10^{-4}$

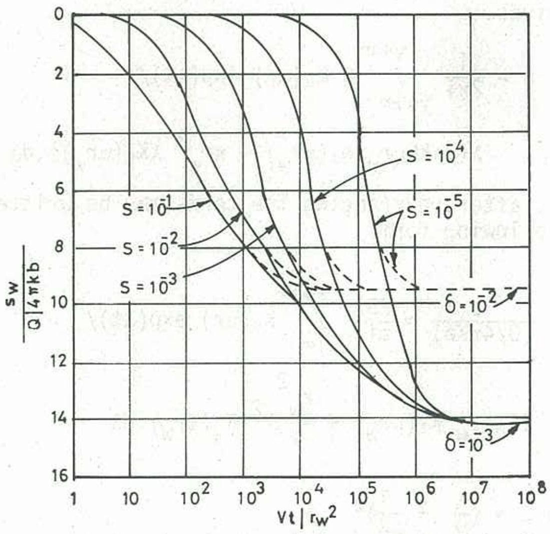

Figure 5 Time - drawdown variation in the well for $\delta = 10^{-3}$ & 10^{-2}

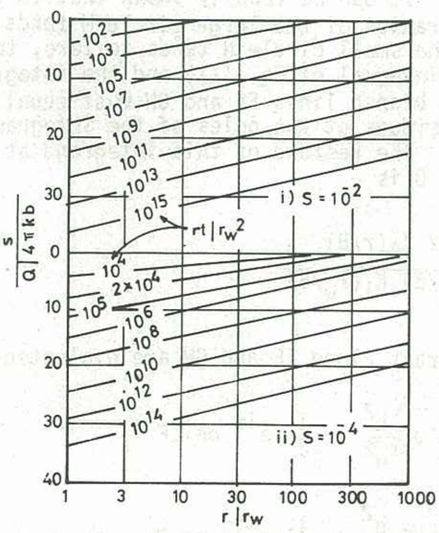

Figure 6 Profiles of piezometric surface for $\delta = 0$

4 CONCLUSIONS

Transient drawdown formula for the problem of a completely penetrating artesian well is developed. The transient equation in which the effects of both well storage and aquitard leakage are incorporated, is seen to offer the best results over the entire range of time.

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