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DESIGN OF SUPERCRITICAL FLOW CONTRACTIONS BY LEAST SQUARE ERROR MINIMIZATION TECHNIQUE

by

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SUMMARY

A brief theoretical background of the supercritical flow contractions together with the existing solution procedures are outlined. Straight wall contractions under the condition of minimum disturbance at the downstream end of contractions are considered. A new computational procedure suitable to study supercritical flow contractions is presented.

The system of seven nonlinear governing equations is reduced to a system of four independent nonlinear equations for four unknowns. The system of four equations is solved by the least square error technique. The error is minimized by using a suitable numerical procedure. The computational procedure is more general in nature and contractions can be designed more effectively for the required contraction ratios once the conditions of approaching flow are known. Design charts showing the variation of different parameters are presented which can be used to evaluate the unknown parameters quickly.

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INTRODUCTION

The design of supercritical flow contractions is basically different from that of subcritical flow contractions due to the reason that supercritical flow results standing waves if there is any change along the course of flow. Ippen and Dawson(1,2) reported pioneering works on the design of supercritical flow contractions. Design for a minimum disturbance is the aim for flow at supercritical velocities so that economical structures can be constructed. It was concluded that the wave heights are determined solely by the deflection angle and straight wall contractions are superior to curved wall contractions as long as the channel bottom is level crosswise and of constant slope in the direction of flow(1). In the analytical procedure, Ippen and Dawson(1,2) first assumed the ratio of final depth to initial depth and adopted a trial and error procedure to evaluate the deflection angle and other unknowns. However, Harrison(3) pointed out that for given values of initial Froude number F_1 and contraction ratio b_3/b_1 (where b_3 and b_1 denote final width and initial width respectively) there is one unique value of deflection angle which defines the required contraction and correspondingly there is one unique value of ratio of final depth to initial depth.

THEORETICAL BACKGROUND

Consider a typical supercritical flow straight wall contraction in a channel of rectangular cross section as shown in Fig.1. Suppose the channel is contracted symmetrically by deflection angle θ . Symmetrical shock waves (which are oblique hydraulic jumps) are propagated from the walls at the entrance of contraction at A and A'. As the flow passes through the positive disturbance(4), the flow direction is changed by angle θ such that the stream lines become parallel near the walls of the contraction, the depth changes from d_1 to d_2 and velocity from V_1 to V_2. The disturbances extend across the channel at angle μ_1 with the upstream channel walls intersecting at point B. The interference produces further disturbances at angle μ_2 with contraction walls. Now, the flow passes through a new field having a new Froude number F_2 in the region ABD and A'BD'. At points C and C' the problem involves expansion of the boundary. Obviously negative wavelets emanate(4,5) at points C and C' and the problem becomes more complicated downstream.

The maximum disturbances result when the intersection of two positive disturbances occurs at the same section where negative disturbances emanate. The downstream disturbances are minimized by designing the contraction such that the second pair of positive disturbances strikes the end of contraction at C and C',thus nullifying the effects with newly emanating negative disturbances as shown in Fig.2.

Considering a section normal to the wave front AB(Fig.2) the depths d_2 and d_1 can be related by the well known hydraulic jump equation (2,6) as

$$\frac{d_2}{d_1} = \frac{1}{2}\left(\sqrt{1 + 8\,F_1^2\,\sin^2\mu_1} - 1\right) \qquad \ldots \qquad (1)$$

Similarly the depths d_3 and d_2 can be related as

$$\frac{d_3}{d_2} = \frac{1}{2}\left(\sqrt{1 + 8\,F_2^2\,\sin^2\mu_2} - 1\right) \qquad \ldots \qquad (2)$$

Since there is no change in momentum parallel to the jump the components of velocity in this direction is same on each side of the jump. Applying the continuity equation perpendicular to the jump the depth ratios can be written (2) as

$$\frac{d_2}{d_1} = \frac{\tan\mu_1}{\tan(\mu_1 - \theta)} \qquad \ldots \qquad (3)$$

$$\frac{d_3}{d_2} = \frac{\tan\mu_2}{\tan(\mu_2 - \theta)} \qquad \ldots \qquad (4)$$

Considering the components of velocity on each side of the jump and applying continuity equation perpendicular to the jump the velocities V_2 and V_1 can be related (3) as

$$V_2^2 = V_1^2\left\{1 - \sin^2\mu_1\left[1 - \left(\frac{d_1}{d_2}\right)^2\right]\right\} \qquad \ldots \qquad (5)$$

Eliminating μ_1, V_1 and V_2 between Eqns.(1)and (5) and representing the velocities in terms of respective Froude numbers the final simplified equation can be written as

$$F_2^2 = \frac{d_1}{d_2}\left[F_1^2 - \frac{1}{2}\,\frac{d_1}{d_2}\left(\frac{d_2}{d_1} - 1\right)\left(\frac{d_2}{d_1} + 1\right)^2\right] \qquad \ldots \qquad (6)$$

Similarly for the exit condition the final equation can be written as

$$F_3^2 = \frac{d_2}{d_3}\left[F_2^2 - \frac{1}{2}\,\frac{d_2}{d_3}\left(\frac{d_3}{d_2} - 1\right)\left(\frac{d_3}{d_2} + 1\right)^2\right] \qquad \ldots \qquad (7)$$

where F_3 is the final Froude number.

By continuity of the flow

$$Q = (b_1 d_1)\, V_1 = (b_3 d_3)\, V_3 \qquad \ldots \qquad (8)$$

Representing Eq.(8) in terms of F_1 and F_3 and simplifying

$$\left(\frac{b_1}{b_3}\right) = \left(\frac{d_3}{d_1}\right)^{3/2}\left(\frac{F_3}{F_1}\right) \qquad \ldots \qquad (9)$$

From geometry the length of contraction L_c can be written as

$$L_c = \frac{b_1 - b_3}{2\tan\theta} \qquad \ldots \qquad (10)$$

In the design of supercritical flow contraction, usually the velocity and depth of the approaching flow and hence the initial froude number F_1 and the required contraction ratio b_3/b_1 are given. It is required to determine the deflection angle θ and flow condition at the contracted channel with minimum disturbance. Since the values of F_1 and b_3/b_1 are usually given there are seven unknowns; viz., θ, μ_1, μ_2, F_2, F_3, d_2/d_1 and d_3/d_2. Harrison(3) has suggested a solution of the nonlinear simultaneous equations namely equations (1) to (4), (6), (7) and (9) for the design of supercritical flow contractions. From inspection of the system of equations it is seen that an exact analytic solution is not feasible because of the nonlinearity in the system. To solve this system of equations Harrison(3) used the Newton-Raphson technique of iterative process. The starting point for the whole solution procedure is an approximate solution for $F_1 = 4$ and $\theta = 10$ degrees obtained by graphical solution proposed by Ippen and Dawson(1,2).

PRESENT STUDY OF THE PROBLEM

In the present work, a simplified system of equations is obtained and a more general method of solution is carried out successfully. The system of above mentioned seven nonlinear equations is reduced to a system of four independent nonlinear equations as shown below.

From Eqn.(1) it is seen that the depth ratio d_2/d_1 is a function of unknown μ_1 and combining the eqn.(1) with eqn.(3).

$$\frac{d_2}{d_1} = \frac{1}{2}\left(\sqrt{1 + 8 F_1^2 \sin^2 \mu_1} - 1\right)$$

$$= f_1(\mu_1) = \frac{\tan \mu_1}{\tan(\mu_1 - \theta)} \quad \ldots \quad (11)$$

Similarly combining Eqs.(2) and (4)

$$\frac{d_3}{d_2} = f_2(\mu_2) = \frac{\tan \mu_2}{\tan(\mu_2 - \theta)} \quad \ldots \quad (12)$$

It is noted that in Eqn.(6) F_2 is a function of d_2/d_1 only and hence using Eqn.(11)

$$F_2 = f_3'\left(\frac{d_2}{d_1}\right) = f_3(\mu_1) \quad \ldots \quad (13)$$

Similarly using Eqns.(7) and (12)

$$F_3 = f_4'\left(\frac{d_3}{d_2}, F_2\right)$$

$$= f_4(\mu_1, \mu_2) \quad \ldots \quad (14)$$

The number of unknowns is now reduced to four, viz., μ_1, μ_2, θ and F_3 and the four independent nonlinear equations can be written as

$$f_1(\mu_1) - \frac{\tan \mu_1}{\tan(\mu_1 - \theta)} = 0 \quad \ldots \quad (15)$$

$$f_2(\mu_2) - \frac{\tan \mu_2}{\tan(\mu_2 - \theta)} = 0 \quad \ldots \quad (16)$$

$$\left[f_1(\mu_1)\, f_2(\mu_2)\right]^{3/2} - \left(\frac{b_1}{b_3}\right)\left(\frac{F_3}{F_1}\right)^{-1} = 0 \quad \ldots \quad (17)$$

$$f_4(\mu_1, \mu_2) - F_3 = 0 \quad \ldots \quad (18)$$

where f_1 , f_2 .... etc. stand for some functions.

It is seen that once the four unknowns μ_1, μ_2, Θ and F_3 are evaluated the other unknowns d_2/d_1, d_3/d_2 and F_2 can be determined by direct substitution.

SOLUTION PROCEDURE

The system of four equations involving Eqns.(15) to (18) is solved by the least square error minimization technique as briefly explained below.

Let the equations be written in terms of the unknowns as

$$f_i(\vec{p}) = 0 \qquad i = 1, 2, \ldots\ldots, n \qquad \ldots \qquad (19)$$

where $\vec{p}$ is an n-dimensional vector of unknowns.

At any point $\vec{p_o}$ the equations may not be satisfied exactly and the errors involved in the equations can be represented as

$$e_i(p_o) = f_i(\vec{p_o}) \qquad i = 1, 2, \ldots n \qquad \ldots \qquad (20)$$

The least square technique aims at determining a vector $\vec{p_m}$ such that

$$E = \sum_{i=1}^{n} e_i^2(p_m) \text{ is a minimum} \qquad \ldots \qquad (21)$$

This is accomplished by a numerical minimization technique namely, the Davidon-Fletcher-Powell variable metric method(7,8).

Let the solution be started from a given point $\vec{p_o}$ which can be represented as

$$\vec{p_o} = \begin{Bmatrix} p_{o_1} \\ p_{o_2} \\ \cdot \\ \cdot \\ \cdot \\ p_{o_n} \end{Bmatrix} \qquad \ldots \qquad (22)$$

where p_{o_1} , p_{o_2} ... etc are the values of the unknowns at the starting point.

A new approximation to the minimum is given by

$$\vec{p}_{j+1} = \vec{p}_j + a_j \vec{w}_j \qquad j = 0, 1 \ldots. \qquad \ldots \qquad (23)$$

where $\vec{w}_j = -(D_j)\,\vec{g}_j \qquad j = 0,1 \ldots. \qquad \ldots \qquad (24)$

$\vec{g}_j$ = gradient of E

$$= \begin{Bmatrix} \dfrac{\partial E}{\partial p_{j_1}} \\ \dfrac{\partial E}{\partial p_{j_2}} \\ \cdot \\ \cdot \\ \dfrac{\partial E}{\partial p_{j_n}} \end{Bmatrix} \qquad \ldots \qquad (25)$$

(D_j) = a symmetric positive definite matrix which is updated after every iteration. (D_o) is taken as identity matrix.

a_j = the direction along $\overrightarrow{w_j}$ which minimizes E in that direction. This is found by Golden section search technique(8)

CONCLUSIONS

In the present study a channel of unit width at the entrance of contraction is considered. The problem is studied for a wide range of F_1 values (between 1.5 and 12.00) and contraction ratios (between 0.10 and 1.00).

Using the least square error technique the unknown parameters for each case are evaluated. The computation is stopped for each case, when the final Froude number becomes unity or approaches unity. The reason is that if $F_3 \leq 1$ the levels in the channel are controlled by conditions further downstream and the developed equations become invalid for this case.

To simplify the design, the results are plotted as shown in Figs.3 to 5. These figures give design parameters for different values of F_1 and contraction ratios. With the help of these figures it is quite easy to evaluate the unknowns once the initial condition of the approaching flow and the required contraction ratio are known. There may be situations where the design has to be done to attain a particular final Froude number F_3. Since the presented procedure is quite general in nature it can be easily modified to meet this situation also.

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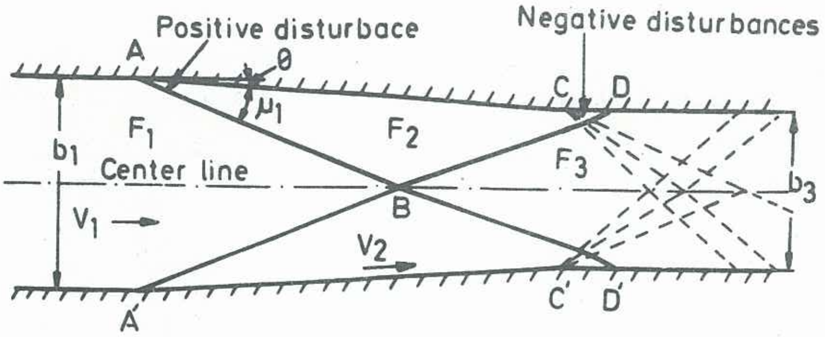

FIG. 1 - STRAIGHT WALL CONTRACTION INVOLVING GENERAL DISTURBANCES

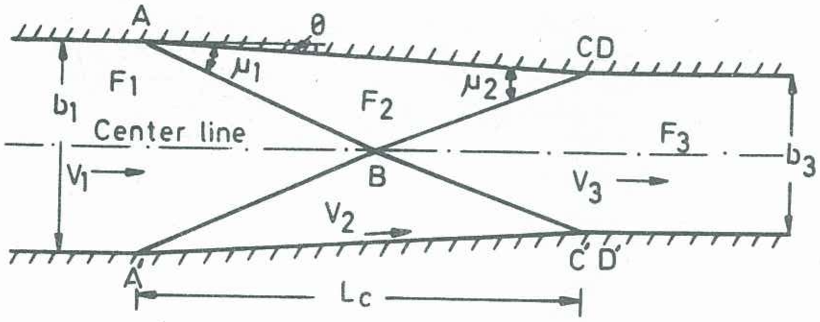

FIG. 2 - STRAIGHT WALL CONTRACTION INVOLVING MINIMUM DOWNSTREAM DISTURBANCE

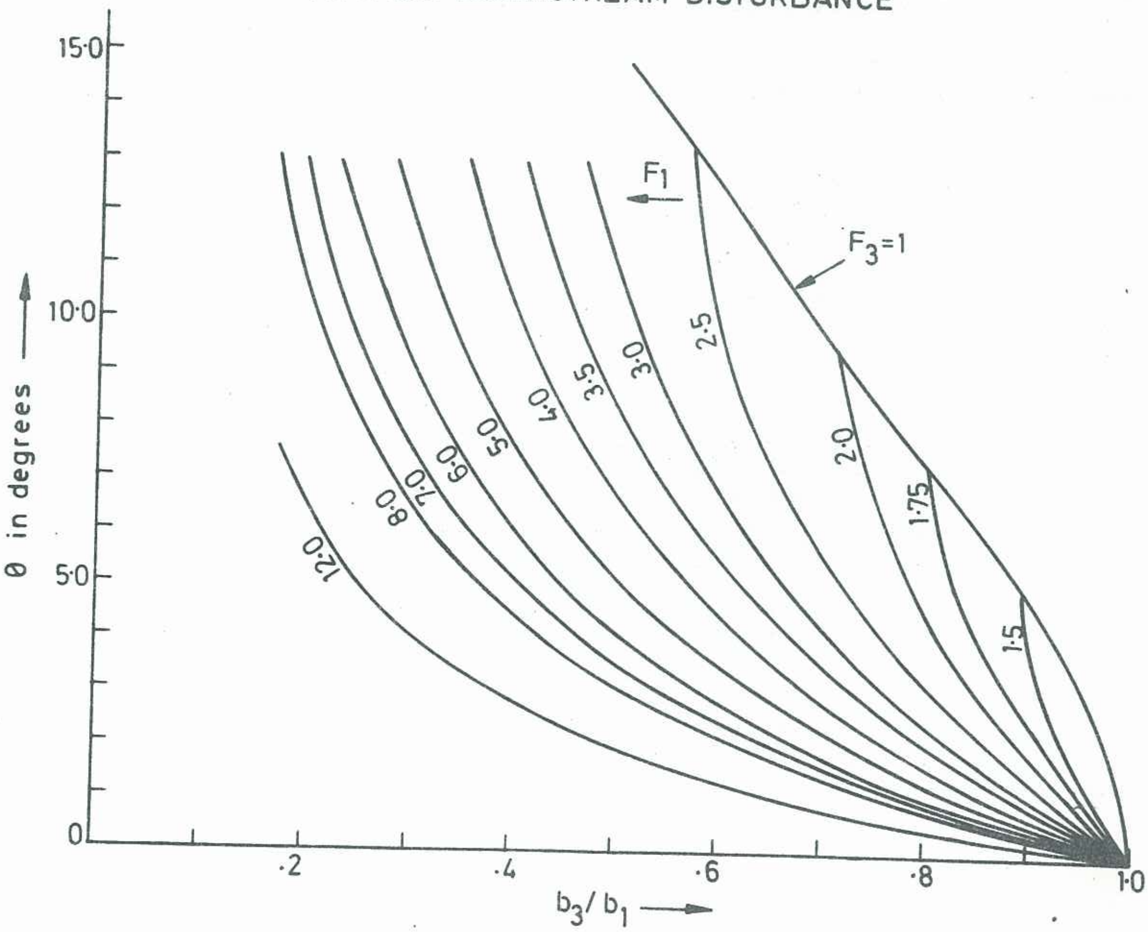

FIG. 3 - PLOT OF b_3/b_1 Vs θ FOR DIFFERENT VALUES OF F_1

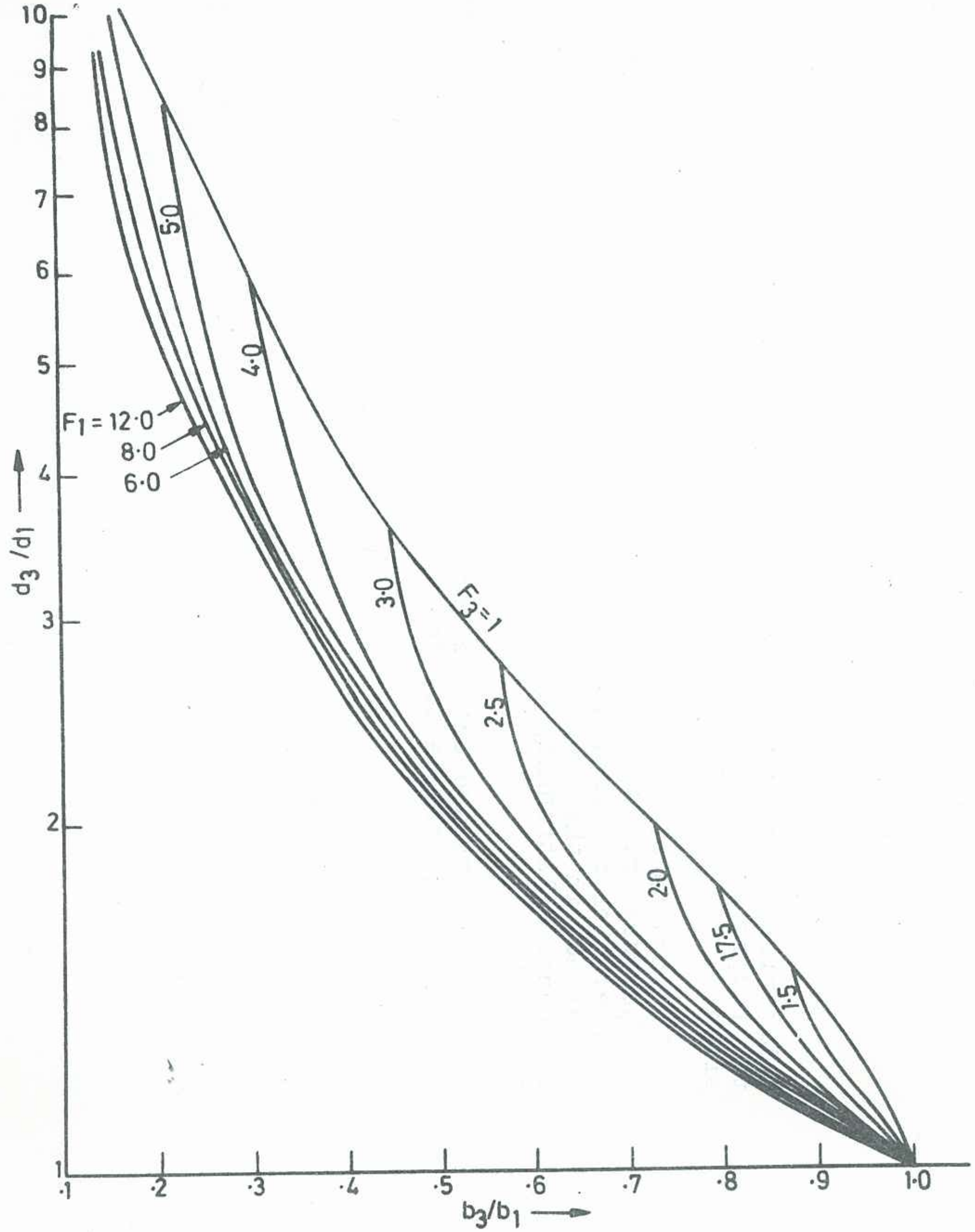

FIG. 4 - PLOT OF b_3/b_1 Vs d_3/d_1 FOR DIFFERENT VALUES OF F_1

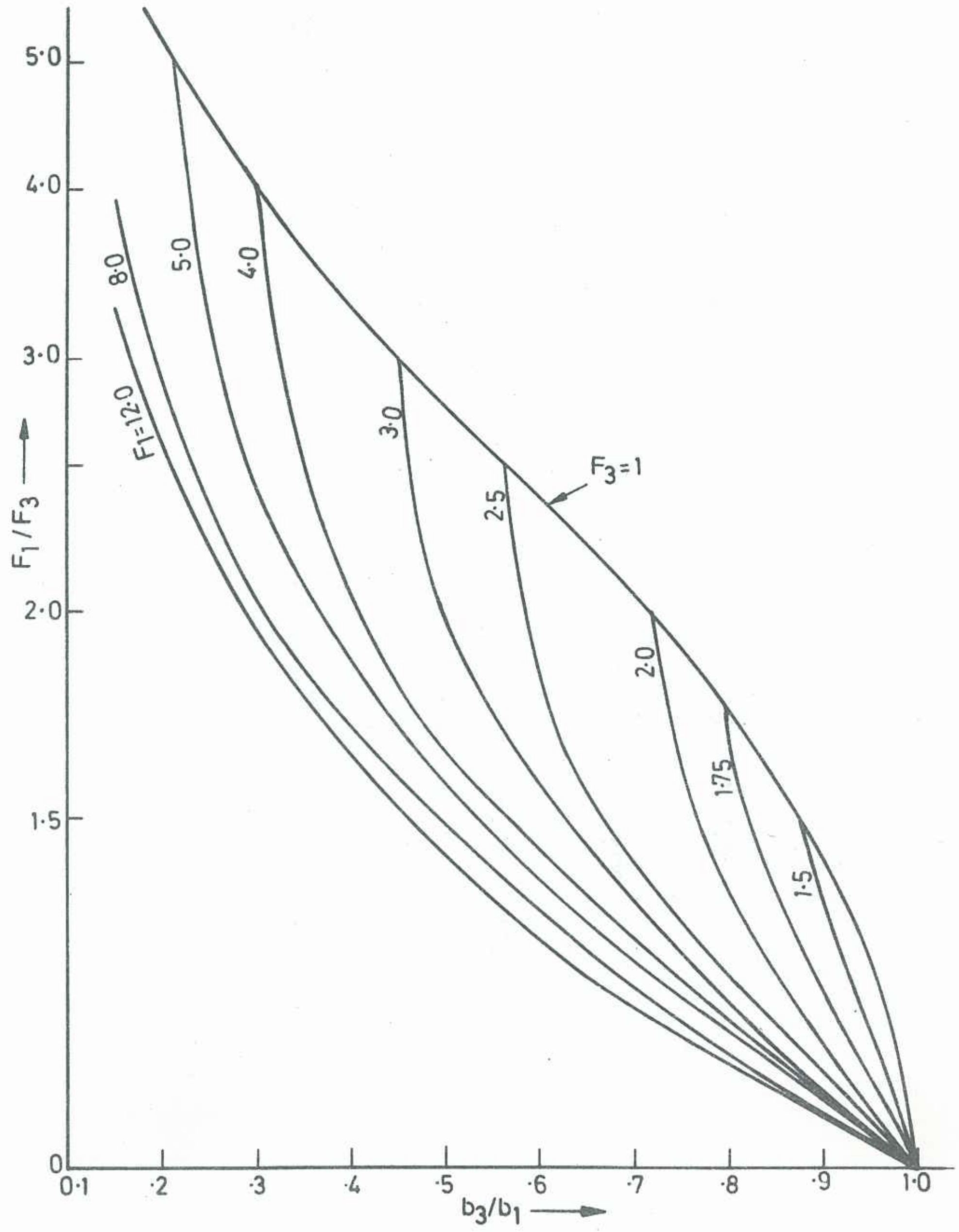

FIG. 5 - PLOT OF b_3/b_1 Vs F_1/F_3 FOR DIFFERENT VALUES OF F_1