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SOLUTION OF FLUID DYNAMIC PROBLEMS BY FINITE ELEMENTS

by

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SUMMARY

The dynamical motion of fluid within a container is modelled by a series of discrete or finite elements. Natural frequencies of oscillations of the system can be obtained by assembling the mass and stiffness characteristics of each element into global matrices and solving the resulting eigenvalue equation. Effects such as surface tension, density variation and non rigid containers with an incompressible fluid have been investigated so far and some interesting results emerge.

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NOMENCLATURE

$\{q\}$ = vector of nodal displacements.
L = Lagrangian, kinetic minus potential energy.
F = dissipation function.
Q = external forcing function.
$[M]$ = mass matrix.
$[H]$ = sloshing matrix.
$[S]$ = stiffness matrix.
$[F]$ = surface tension matrix.
$[K]$ = local stiffness matrix.
$[T]$ = incompressibility influence matrix.
$[\phi]$ = zero frequency influence matrix.
ρ = density.
g = gravity.
χ = surface tension coefficient.
V = volume.

Subscripts

f = fluid.
t = tank.
r = reduced.
s = further reduced.
i,j = nodal locations.

Superscripts

e = element.
' = global.
T = transpose.

1. INTRODUCTION

Some recent publications (1) (2) have proposed an outline of a technique for the vibration analysis of fluids sloshing in tanks. Problems involving the harmonic oscillation of fluids occur in many areas, tidal motion, dynamic mixing of fluids in distillation processes, bulk transportation of fluids, liquid fuel in rockets are but a few. Generally the geometry of these situations excludes exact analysis and thus the use of a finite element technique would appear to be particularly appropriate.

In this work Lagrange's equation is used as a basis to obtain a standard vibration equation from which eigenvalues and vectors, pertaining to various natural modes of surface motion, can be extracted. In this vibration equation the mass matrix relates to a series of assembled finite elements which have their mass lumped at nodes corresponding to the degrees of freedom of the element. If the walls of the tank are elastic then a wall node would have its mass altered to accommodate this. The "stiffness" matrix is composed of several effects, namely; changes in surface level are given in a sloshing matrix, surface tension in a surface tension matrix, strain energy in fluid in a fluid stiffness matrix and of the tank in a tank stiffness matrix.

Results of sloshing frequencies are presented and commented upon for several combinations of the above mentioned quantities, others still require further study.

2. THEORY AND FINITE ELEMENT MODEL

The following development of the technique is expressed in the usual finite element notation and presumes use of assumed displacement fields. Lagrange's equation is used as a basis for the derivation of the vibration equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial q_i} = Q_i \qquad 1$$

In order to determine natural frequencies of oscillation F and Q_i may be set to zero. L can be defined for the whole region or since it is a scalar quantity for a finite element of a region.

L = kinetic energy - potential energy of surface motion - potential energy due to surface tension - strain energy of fluid - strain energy of container.

When L is substituted into (1) and equation then solved, a standard vibration equation is evolved.

$$\left[[M]_f + [M]_t\right]\{\ddot{q}\} + \left[[H] + [F] + [K]_f + [K]_t\right]\{q\} = 0, \qquad 2a$$

$$\text{i.e.} \qquad [M']\{\ddot{q}\} + [S']\{q\} = 0 \qquad 2b$$

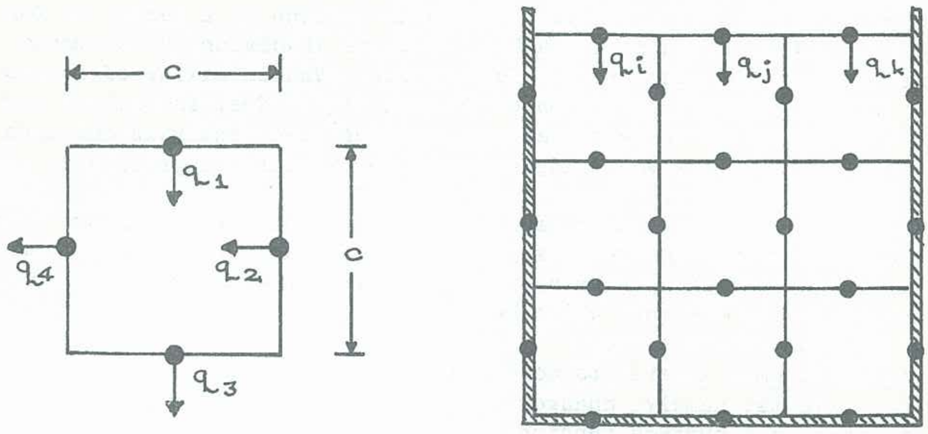
The various matrices in (2a) are assembled by the direct stiffness method of structural analysis from the element component matrices.

$$[M]_f = \sum_{\text{elements}} [M]^e_f \ , \quad \text{etc.} \qquad 3$$

Only the surface tension matrix [F] has a different derivation as will be shown later.

With the absence of any viscous dissipation or structural damping the statement of the problem as in equation (2) corresponds to the Principle of Virtual work where the assumed displacement field {q} is essentially a set of compatible virtual displacements. Thus it is seen that equation (2) represents an extreamum principle and the theory of the finite element method in general (3) indicates that in the limiting case of an infinite number of elements, convergence to an 'exact' solution is ensured. It is now in order to proceed to the presentation of the finite or discrete element used and the derivation of the elemental properties.

2.1 THE FINITE ELEMENT. As the technique presented herein is still at a partially embryonic stage it is expedient to use a simple element in order to assess the potential of the scheme. For this reason the analysis is restricted to two dimensions with unit thickness and the element used is a square of side c, figure 1a.



A) The finite element. B) 3 x 3 grid in tank.

Figure 1

At the mid point of each side is a node with a single degree of freedom at which the effective mass corresponding to that degree of freedom is lumped.

2.2 MASS MATRIX. Since there is only a single d.o.f. associated with each node then vertical motion is reflected by nodes 1 and 3, and horizontal motion by nodes 2 and 4. Thus the effective nodal mass from a vibrating point of view is $\rho c^2/2$. There is no cross coupling between masses and thus,

$$[M]^e_f = \frac{\rho c^2}{2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} . \qquad 4$$

For the container the mass of any wall nodes can be added to by an amount equal to the mass per unit length of the wall times c.

2.3 SLOSHING MATRIX. In a U tube manometer the change in potential is the weight per unit length times the change in height and is independent of the shape of the tube. For unit displacement the force required is the weight per unit length. Consequently in this model, the stiffness associated with surface motion in an element is merely the weight per unit length ρgc, and it only affects surface nodes.

$$[H]^e = \rho gc\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \qquad 5$$

2.4 SURFACE TENSION MATRIX. Surface tension is defined as the force per unit length on the surface. If the surface is disturbed then surface tension produces restoring forces. In this elemental approach if one surface node is given unit displacement then its neighbouring undisturbed elements experience forces equal and opposite. Therefore it is impossible to define an element surface tension matrix uniquely and two elements, at least, have to be lumped together. In reality the virtual displacements are small and simple statics may be used to show that the surface tension matrix linking nodes i and j in figure 1b can be presented as,

$$[F]^e_{ij} = \frac{\chi}{c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad 6$$

and if all three surface nodes are assembled then the total surface tension matrix for this problem would be,

$$[F] = \frac{\chi}{c} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \qquad 7$$

2.5 FLUID STIFFNESS MATRIX. This corresponds to the structural stiffness matrix where the coefficient K^e_{ij} is the force at i required to give unit displacement at j, and can be obtained from the unit displacement theorem of structural analysis.

$$K^e_{ij} = \int_V \{\varepsilon_i\}^T\{\sigma_j\}\, dV. \qquad 8$$

In this $\{\varepsilon_i\}$ is a vector of strains due to unit displacement at i and $\{\sigma_j\}$ a vector of stresses due to unit displacement at j. In the element model employed herein the volumetric strain-displacement relationship is,

$$\{\varepsilon\} = [D]\{q\} = \frac{1}{c}\,[q_1 + q_2 - q_3 - q_4], \qquad 9$$

where $[D] = \frac{1}{c}\,[1 \quad 1 \quad -1 \quad -1]$ is the strain - displacement influence matrix.

$$\{\sigma\} = [B]\{\varepsilon\}, \qquad 10$$

where $[B]$ is the elasticity matrix and is in fact the bulk modulus for the fluid, call this B.

Introducing these concepts into (8) and since neither $\{\varepsilon_i\}$ or $\{\sigma_j\}$ involve field variables because of the simple displacement field, the fluid element stiffness is found to be,

$$[K]^e_f = \int_V [D]^T B [D]\, dV,$$

$$= B \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}. \qquad 11$$

2.6 TANK STIFFNESS MATRIX. This will depend totally upon the type of structure used and will pertain only to wall nodes. The unit displacement theorem may again be used to determine the coefficients and these can be assembled into the global stiffness matrix in the appropriate locations.

3. EFFECT OF INCOMPRESSIBILITY AND ZERO FREQUENCY MODES

For an element the incompressibility constraint condition can be expressed, from equation (9), as,

$$\{\varepsilon\} = \frac{1}{c}\,[q_1 + q_2 - q_3 - q_4] = 0.$$

Each element contributes such a condition and these can be used to reduce the number of degrees of freedom from $\{q\}$ to $\{q_r\}$ via an incompressibility constraint influence matrix $[T]$ where

$$\{q\} = [T]\{q_r\} . \qquad 12$$

Substitute for $\{q\}$ in equation (2b) and premultiply by $[T]^T$ and thereby obtain a reduced form of the vibration equation.

$$[M_r']\{\ddot{q}_r\} + [S_r']\{q_r\} = 0 , \qquad 13$$

where
$$[M_r'] = [T]^T[M'][T]$$
$$[S_r'] = [T]^T[S'][T] .$$

The matrix $[S_r']$ is generally highly singular due to the absence of stiffness associated with many nodes when the fluid is incompressible. This signifies that many modes of vibration have zero sloshing frequency and these modes may be eliminated before the non zero frequencies are evaluated.

Let $\{q_r\} = \{q_z \mid q_s\} = [\phi]\{q_z\}$, where $\{q_z\}$ are the d.o.f. with zero sloshing frequency and $[\phi]$ is an influence matrix for zero frequency modes. Substitute for $\{q_r\}$ in equation (13).

$$[M_r']\left[[\ddot{\phi}]\{\ddot{q}_z\}\right] + [S_r']\left[[\phi]\{q_z\}\right] = 0.$$

Thus if $\{\ddot{q}_z\} = 0$ then $[S_r'][\phi] = 0 = [\phi]^T[S_r']$. Multiply equation (13) by $[\phi]^T$.

$$[\phi]^T[M_r']\{\ddot{q}_r\} + [\phi]^T[S_r']\{q_r\} = 0 ,$$

i.e.
$$[\phi]^T[M_r']\{\ddot{q}_r\} = 0.$$

Integrate twice yielding $[\phi]^T[M_r']\{q_r\} = Et + F$, where E must be zero otherwise $\{q_r\}$ would increase linearly with time and F may be taken zero as a suitable arbitrary origin for $\{q_r\}$.

Let
$$[\phi]^T[M_r'] = [P \mid R],$$

thus
$$[P \mid R]\{q_z \mid q_s\} = 0,$$

and so
$$\{q_z\} = -[P]^{-1}[R]\{q_s\} ,$$

thus finally

$$\{q_r\} = \{q_z \mid q_s\} = \begin{bmatrix} -[P]^{-1}[R] \\ ---- \\ I \end{bmatrix}\{q_s\} = [D]\{q_s\} \qquad 14$$

Substitute for $\{q_r\}$ from equation (14) into equation (13) and premultiply by $[D]^T$.

$$[M_s']\{\ddot{q}_s\} + [S_s']\{q_s\} = 0, \qquad 15$$

where
$$[M_s'] = [D]^T[M_r'][D]$$
$$[S_s'] = [D]^T[S_r'][D] .$$

This may now be reduced to an eigenvalue equation as follows. At a resonant frequency the motion will be harmonic,

$$\{q_s\} = \{q_{s0}\}\sin \omega t,$$

and so

$$\left[-\omega^2 [M_s'] + [S_s']\right] \{q_{s0}\} = 0. \qquad 16$$

In this $[M_s']$ is a symmetric matrix and can be reduced to $[M_s'] = [L][L]^T$, where $[L]$ is an upper triangular matrix. Substitute for $[M_s']$ and divide throughout by $[L]$

$$\left[[L]^{-1}[S_s'] - \omega^2 [L]^T\right]\{q_{s0}\} = 0$$

i.e.

$$\left[[W] - \omega^2 [I]\right]\{Q_{s0}\} = 0 \qquad 17$$

$$\text{where} \quad [W] = [L]^{-1}[S_s'][L]^{-1T}$$

$$\{Q_{s0}\} = [L]^T \{q_{s0}\} .$$

Any standard eigenvalue technique can be used to evaluate the sloshing frequencies since $[W]$ is a symmetric matrix.

4. SOME ILLUSTRATIVE RESULTS

4.1 FLUID INCOMPRESSIBLE, TANK RIGID, NO SURFACE TENSION. For this case it is possible to compare the non-zero sloshing frequencies with analytical values derived by Lamb (4), given by,

$$\omega = \left[\frac{2\pi g}{\lambda} \tan h \left(\frac{2\pi h}{\lambda}\right)\right]^{\frac{1}{2}} , \qquad 18$$

where λ is the wavelength of the surface motion ($\lambda = 2b$ for first natural frequency, $\lambda = b$ for second, etc.) and h and b are the height and width of the tank. For various grid sizes and tank aspect ratios the results are presented in the table below.

GRID SIZE		$\omega_1 / (g/b)^{\frac{1}{2}}$			$\omega_2 / (g/b)^{\frac{1}{2}}$		
DIVISIONS HIGH	DIVISIONS LONG	COMPUTED	ANALYSIS	% ERROR	COMPUTED	ANALYSIS	% ERROR
3	3	1.6329	1.7691	7.7	1.9817	2.5066	20.9
4	4	1.6866	1.7691	4.6	2.1490	2.5066	14.2
5	5	1.7143	1.7691	3.1	2.2510	2.5066	10.2
6	6	1.7301	1.7691	2.2	2.3165	2.5066	7.6
7	7	1.7401	1.7691	1.6	2.3605	2.5066	5.8
6	4	1.6906	1.7720	4.6	2.1491	2.5066	14.2
4	6	1.7053	1.7566	2.9	2.3155	2.5064	7.6
4	8	1.6733	1.6974	1.4	2.3853	2.5019	4.6

High errors at small numbers of elements and also at higher modes than the first are expected due to the simplicity and small number of elements being unable to represent the displacements associated with the modes to any degree of accuracy. At larger grids the relative errors indicate that the limiting case of an infinite number should give the exact solution. It is considered unnecessary to investigate larger and larger grids.

By a simple alteration to the input data for the computer programme which solves the sloshing model, relative densities between horizontal layers on the grid can be imposed. Using the 6×6 grid, several permutations of density variation were studied and the following comments may be made from the results.

With a density ratio of two starting at the bottom layer and moving upwards there was no appreciable change in the sloshing frequency until the ratio existed between the top layer and the rest. Here the reduction in $\omega_1{}^2$ was 20% and in $\omega_2{}^2$, 8%. However the trend of the results seems to indicate that as the lower density region occupies a smaller and smaller fraction of the total depth, then there would be a significant reduction in frequency and could therefore account for the oil on water phenomenon.

Taking the top layer ($1/6^{th}$ of total) at unit density and the remainder at an increasingly higher density gives the result that as the density ratio increases the sloshing frequencies of surface motion decay exponentially to values which would have been obtained had the bottom high density region, in fact, been solid.

4.2 SURFACE TENSION INCLUDED. With the 3×3 grid several solutions were obtained at various values of the surface tension coefficient and the following results emerged.

$$\omega_1 = 1.6329\left[\frac{g}{b}\left(1 + \frac{9\chi}{\rho g b^2}\right)\right]^{\frac{1}{2}}$$

$$\omega_2 = 1.9817\left[\frac{g}{b}\left(1 + \frac{27\chi}{\rho g b^2}\right)\right]^{\frac{1}{2}} .$$

However $\chi/\rho g b^2$ is, with water under air and b equal to one meter, equal to 73×10^{-7}, and it would be an exceptional physical situation whereby the surface tension would affect the sloshing frequency.

4.3 FLUID INCOMPRESSIBLE, TANK ELASTIC, NO SURFACE TENSION. In order to have a reasonably simple tank stiffness matrix the walls of the tank will be considered to be cantilevered from a rigid base. Thus the unit displacement theorem can again be used to give the coefficients in the stiffness matrix as,

$$K_{ij,t} = \frac{9EI}{a_j a_i^3}\left[a_i - \tfrac{1}{2}(a_i + a_j) + \frac{a_j}{3}\right] \qquad 19$$

where $i \geqslant j$, $a_i = c(i - \frac{1}{2})$, E = Young's modulus and I = second moment of area per unit width.

Consider the simple 2×2 grid with ten degrees of freedom, figure 2a. The stiffness matrices for both walls would be,

$$[K]_t = \frac{EI}{c^3}\begin{bmatrix} 24 & 32/9 \\ 32/9 & 8/9 \end{bmatrix} ,$$

and the mass matrix for each wall would be,

$$[M]_t = \rho_t ct \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where t is the thickness of the wall. These can be assembled into the global mass and stiffness matrices for the whole system at the appropriate locations. Using the incompressibility constraint conditions to eliminate the degrees of freedom associated with nodes 3,4,7 and 8, the reduced mass and stiffness matrices associated with nodes 1,2,5,6,9 and 10 respectively, are as follows.

$$[M_r'] = \frac{\rho c^2}{2}\begin{bmatrix} 4 & 1 & -3 & -1 & 0 & 0 \\ 1 & 2 & -1 & -1 & 0 & 0 \\ -3 & -1 & 8 & 2 & -3 & -1 \\ -1 & -1 & 2 & 4 & -1 & -1 \\ 0 & 0 & -3 & -1 & 4 & 1 \\ 0 & 0 & -1 & -1 & 1 & 2 \end{bmatrix} + \rho_t ct \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[S_r'] = \rho g c \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} + \frac{24EI}{c^3}\begin{bmatrix} 1 & 4/27 & 0 & 0 & 0 & 0 \\ 4/27 & 1/27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4/27 \\ 0 & 0 & 0 & 0 & 4/27 & 1/27 \end{bmatrix}$$

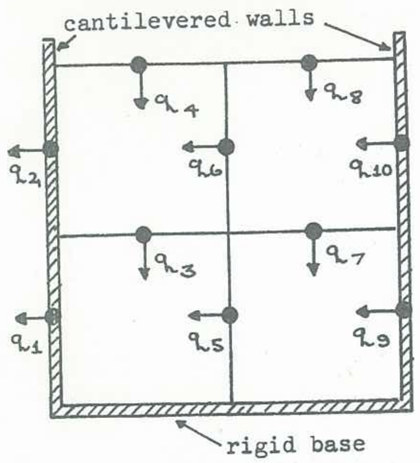


Figure 2A) 2 x 2 grid in elastic tank.

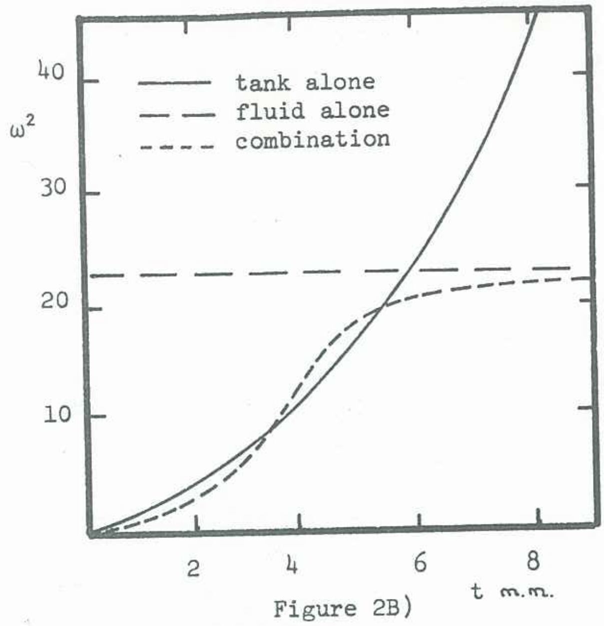


Figure 2B)

If the fluid is taken as water and the tank of steel of overall dimensions 1 × 1 meter then a series of eigenvalues can be extracted from the above matrices for a range of values of wall thickness. When only the first eigenvalue is plotted against wall thickness the curve shown in figure 2b is obtained. Also plotted on this graph is the first natural frequency of the fluid alone and the first natural frequency of the wall alone. These latter two quantities are based on the modelled form of the system not the exact form.

From figure 2b it is seen that the first natural frequency of the combined system follows that of either the wall or the fluid, whichever is the lowest. Thus the mode of the lowest frequency of the combined system changes from a wall mode with no sloshing up to the tuned frequency ($\omega_t = \omega_f$) and thereafter continues as a sloshing mode (no wall motion). Clearly in the actual system other modal interactions occur but the simplicity of the model used herein excludes their detection and further study is required.

5. CONCLUSION

The results obtained so far, indicate that the modelling of a region of fluid in a container subject to dynamic motion of itself and its boundaries and be effectively achieved by the use of a discrete element technique. Clearly further work is required to generalise the element used and the geometry of the regions and the containers. Once this has been done and the constraint of incompressibility removed the solution technique, though now much more complex, can be applied to almost any problem involving dynamic oscillation of fluids.

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