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THE DIFFUSIVE INTERFACE IN DOUBLE DIFFUSIVE CONVECTION

by

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S U M M A R Y

Layered convection occurring under the influence of two diffusing components, for example heat and salt, has been observed many times in the laboratory, in lakes, and in the ocean. However while the fluxes of the two components have been measured as functions of appropriate parameters, no theory has been forthcoming which fully explains the experimental results. This paper deals with the particular case of the diffusive interface, which is produced when the motion is driven by the more rapidly diffusing of the components. By analogy with ordinary one-component convection, and by means of an argument to explain the ratio of the two fluxes which is based on observations of the flow, the determination of the flux of the driving component can be reduced to the problem of solving a one-dimensional diffusion problem.

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Introduction

Free thermal convection in a horizontal layer of homogeneous fluid is a well-known and quite thoroughly understood phenomenon. Motion is induced by buoyancy forces produced by thermal expansion of the fluid, and limited by both the viscosity and the thermal diffusivity of the fluid. There are two controlling parameters, namely the Rayleigh number Ra

$$Ra = \frac{g\alpha\Delta T h^3}{\nu K_T}$$

and the Prandtl number

$$Pr = \frac{\nu}{K_T}$$

Here g is the acceleration due to gravity, $\alpha\Delta T$ the fractional density difference between top and bottom of the layer, h the depth of the layer, ν the kinematic viscosity and K_T the thermal diffusivity.

There are two characteristics of this type of convection which are important here. Firstly the motion extends throughout the whole depth h. Secondly at high values of Ra and Pr (a common case in practice) much of the fluid is nearly isothermal, most of the temperature change occurring in the boundary layers. These boundary layers have been shown by Howard[1] to be well represented by a model wherein long quiescent periods, during which fluid near the boundary is stationary and heated (or cooled) by conduction, alternate with short bursts of motion during which the heated (cooled) fluid is removed from the boundary.

In a number of convective situations in liquids, this picture is substantially modified by the presence of a second diffusing component which, like the temperature, is not uniformly distributed. Provided the two components have different diffusivities in the liquid then it becomes possible for the liquid to be divided into a number of layers, each layer being less dense than the one beneath it, but substantially homogeneous within itself. The most important examples of this "double-diffusive" convection (DDC) have been seen in the ocean, where the two components are heat and salt. Lake Vanda, Antarctica, is another case of heat and salt, but the phenomenon has also been observed in the laboratory using two solutes (e.g. salt and sugar) in water, and again in solidifying metal alloys. The present state of knowledge has been surveyed by Turner[2].

In this type of convection one component must be distributed in a gravitationally-destabilizing manner so as to provide energy for the motion, while the other component is gravitationally stabilizing so as to maintain the interfaces between the layers. The former may be called the driving component, and the form of the interfaces depends on whether it is the faster-diffusing (T) or slower-diffusing (S) component. The latter case leads to the "finger" interface within which there is a vertical flow from one layer to the next driven by horizontal diffusion between counter-flowing cells. In the former case there is in general no flow across the interface, and as the transport of both components across the interface is by means of molecular diffusion, this is known as the "diffusive" interface. It is with this type of interface that we are concerned here.

To describe DDC, we clearly need two parameters more than for one-component ("thermal") convection, since we must allow for the effects firstly of the distribution of concentration of the second component and secondly of its different diffusivity. These effects are conventionally allowed for by introducing the two new parameters

$$R_\rho = \frac{\beta\Delta S}{\alpha\Delta T} \quad \text{(density anomaly ratio)}$$

and

$$\tau = \frac{K_S}{K_T} \quad \text{(diffusivity ratio)}.$$

$\beta\Delta S$ and $\alpha\Delta T$ are now the individual contributions of the S and T components to the fractional density difference between two adjacent layers. In the diffusive interface case, where T is the driving component, it follows that $R_\rho \geq 1$ in order for the upper layer to be less dense than the lower.

Any theory of the diffusive interface must account for its remarkably simple flux properties. Fluxes of both components have been measured by Turner[3] for $\tau = 0.02$ (heat and salt), and by Shirtcliffe[4] for $\tau = 0.33$ (salt and sugar). Apart from the region $1 < R_\rho \lesssim 2$ in the heat-salt case, where the results seem to be affected by entrainment across the interface, both sets of results are consistent with the relationships

$$F_S/F_T = \tau^{1/2} \tag{1}$$

and

$$F_T^* = A R_\rho^{-B} \tag{2}$$

where $B = 1.8$ for $\tau = .02$, $B = 12.6$ for $\tau = .33$, and $A = 2.6$ in both cases. In (1), F_S and F_T are the fluxes of the S and T components respectively across the interface, measured in the units of mass per unit area per unit time. In (2), F_T is non-dimensionalized by writing

$$F_T = F_P F_T^*$$

where F_P is the flux which would be produced through a layer of the same depth bounded above and below by solid planes whose temperature differs by ΔT. That is, F_T^* shows the extent to which the density effect of the S component modifies ordinary thermal convection.

It has been pointed out by Rooth (see Veronis(5)) that (1) points to an intermittency of the diffusive interface similar to that exhibited by the boundary layers in thermal convection, and the argument has been taken further by Turner(2). At the beginning of a quiescent period there are sharp steps $\alpha\Delta T_*$ and $\beta\Delta S_*$ in the two components on both sides of the interface. These then produce, by molecular diffusion, boundary layers of S and T whose thicknesses are proportional to $K_S^{\frac{1}{2}}$ and $K_T^{\frac{1}{2}}$ respectively. After some time t_* these boundary layers will be swept away by further eddies, so that the fluxes F_S and F_T averaged over the period t_* are in the ratio

$$\frac{F_S}{F_T} = \frac{\beta\Delta S_*}{\alpha\Delta T_*} \tau^{\frac{1}{2}} \tag{3}$$

In order for (3) to agree with (1), we need to understand why

$$\beta\Delta S_* = \alpha\Delta T_* \tag{4}$$

It may readily be seen that at $t = t_*$, because T is both the more rapidly diffusing and the destabilizing component, there is a thin sub-layer on the outside surface of the interface (i.e. above and below) which is buoyant. The significance of (4) is that the eddies remove only this buoyant sub-layer, and do not penetrate in to the stably stratified interior of the interface. The aim of the present paper is to show that this model of the diffusive interface can be extended in a consistent fashion, firstly to explain why only the buoyant sub-layer is removed by the large-scale intermittent motion, thus providing an understanding of (1), and secondly allowing a calculation of the flux F_T^* for comparison with the observation (2).

Flow near the interface

Observations of the diffusive interface show that liquid flows away from the interface in the form of narrow plumes. These are almost two-dimensional line plumes very near the interface, with a rather random pattern of such line plumes enclosing polygonal areas of the interface within which there is a return flow directed towards the interface. The plumes are clearly driven by a buoyancy force, which must originate in the buoyant sub-layer which is produced on the outside surfaces of the interface by differential diffusion (see figure 4). Thus we must look at the return flow to see how it transports liquid along the outer surfaces of the interface into the plumes. As we shall see, there are two properties of this flow which ensure that the interfacial liquid is fed into the plumes "from the outside in"; that is, the outer surfaces of the interface contribute preferentially to the flow into the plumes. (4) follows from this property, since as soon as non-buoyant liquid is supplied to the plumes their driving force vanishes and the whole flow, including that which transports interfacial liquid sideways into the plumes, must stop, thereby initiating a further quiescent period.

Observation suggests that an individual "eddy", that is a cell consisting of an approximately axisymmetric flow towards a stagnation point on the interface together with the plumes that form its lateral boundaries, lasts for a time at least comparable with that taken by a typical particle to complete an orbit of the cell. Thus we may regard the "external" flow as being quasi-steady after a short initial period of acceleration. This flow must match an "internal" flow within the initially static interfacial liquid which has a velocity profile of the form indicated in figure 1.
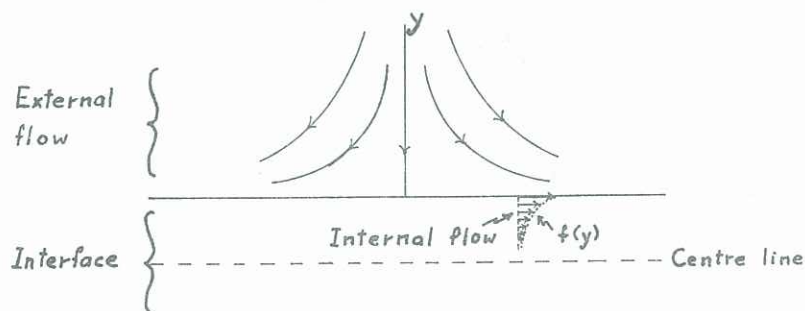

FIGURE 1   External and internal flows.

The external flow indicated in figure 1 has a pressure maximum at the origin which must be supported by the boundary. The interface can provide the required horizontal pressure gradient by deforming, since it contains a substantial vertical density gradient. Thus the isopycnals may take the form sketched in figure 2, but the internal flow for which we shall look will be only slightly modified to flow along the isopycnals instead of horizontally.
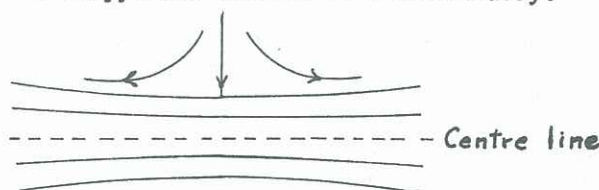

FIGURE 2   Isopycnals within the interface.

The equation of motion of the liquid can be written in terms of the vorticity $\underset{\sim}{\omega}$, bearing in mind that the vortex lines are not stretched in axisymmetric flow, as

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + (\underset{\sim}{u}.\nabla)\underset{\sim}{\omega} = \nu\nabla^2\underset{\sim}{\omega} - \frac{1}{\rho^2}\nabla p \times \nabla \rho \tag{5}$$

Fluid within the interface is thus accelerated by two effects given by the two terms on the right of this equation. The first of these terms relates to the diffusion of vorticity (or momentum), and we shall first look at this process in relation to our internal flow.

Let the internal flow profile have a vertical length scale ℓ, and let the horizontal velocity gradient at $y = 0$, the boundary between internal and external flows, be $\left(\frac{du}{dx}\right)_0 = k$. In keeping with the quasi-steady nature of the external flow, we shall assume that the profile of the internal flow is of constant form. However we note that as $k > 0$, the horizontal velocity at a plane $x = \xi + d\xi$ exceeds that at $x = \xi$, so that there is a net outflow from between the two planes. Continuity of volume requires that this be replaced by a vertical inflow, which can only come from a penetration of the external flow region into space formerly occupied by the outer part of the internal flow. That is, the internal flow profile propagates into the interface at some speed v as liquid entrained viscously by the external flow is carried off horizontally.

Continuity then requires that we write approximately

$$v = \ell k.$$

Also we can write

$$\omega = \frac{\partial u}{\partial y}$$

and

$$\frac{\partial u}{\partial t} = v\frac{\partial u}{\partial y}.$$

Then if we assume that the external flow is approximately irrotational so that

$$u = kx \quad \text{at} \quad y = 0,$$

we can write generally that

$$u = kxf(y+vt)$$

with

$$f(y+vt) = 1 \quad \text{at} \quad y+vt = 0,$$

and

$$f(y+vt) \rightarrow 0 \quad \text{as} \quad y+vt \rightarrow -\infty.$$

It follows that we can write (5) (neglecting the second term on the right) as

$$v\frac{df}{dy} + kf^2 = \nu\frac{d^2f}{dy^2}$$

or, since

$$\frac{df}{dy} \sim \frac{f}{\ell} \quad \text{etc.,}$$

$$k(1+f) \sim \frac{\nu}{\ell^2}$$

It follows that, near $y+vt = 0$, $\ell \sim \left(\frac{\nu}{2k}\right)^{\frac{1}{2}}$

while at $(y+vt) \rightarrow -\infty$ $\ell \sim \left(\frac{\nu}{k}\right)^{\frac{1}{2}}$

Observations of the diffusive interface show that the interface is very thin when $R_\rho \div 1$, and that in these conditions there is considerable symmetry between the flows above and below the

interface. This is consistent with the above analysis, since under these conditions $\ell \gg \varepsilon$ where ε is the interface thickness, so that the two sides of the interface are viscously coupled. It is quite possible that the flux ratio departs from (1) here, but by only a small amount; when $R_\rho = 1$, the interface is apparently of zero thickness at the start of a quiescent period, so that $\beta\Delta S_* = \frac{1}{2}\,\beta\Delta S$, $\alpha\Delta T_* = \frac{1}{2}\,\alpha\Delta T$, and $\dfrac{\beta\Delta S_*}{\alpha\Delta T_*} = 1$ necessarily.

When the interface is thicker ($\varepsilon \doteq 0.2$ cm for the salt/sugar case) the two sides of the interface are observed to decouple, and approximate calculations suggest that indeed $\ell \sim \varepsilon$ at this stage.

For very thick interfaces it seems, although more measurements are needed to establish it, that $\ell < \varepsilon$. Thus here a burst of motion occupying a time interval short compared to t_* could indeed skim off just the outer, buoyant, sub-layer of the interface into the plumes. However here a second effect occurs which is probably also important, related to the accelerations associated with the second term on the right of (5). The buoyant sub-layer contains a density extremum, and figure 3 shows the senses in which $\underset{\sim}{\omega}$ is caused to increase by this term, above and below this extremum.
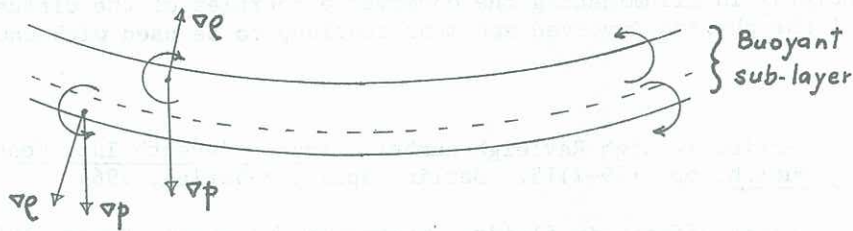

FIGURE 3

It is easily seen that the effect here is to accelerate the buoyant liquid towards the adjacent plume, whilst rotating the stable interior of the interface back towards its state of equilibrium. Thus this gravitational process reinforces the diffusive one, and both combine to peel preferentially the buoyant sub-layer off the interface, the gravitational process also acting positively to restrain the stable interfacial liquid from being entrained into the external flow. (4) follows from this as explained previously, showing (3) to be consistent with (1).

The convection as a diffusion problem

Our model of the diffusive interface envisages periods t_* of differential diffusion separated by relatively short bursts of motion which skim the diffusive boundary layers off the interface. It is possible therefore to concentrate entirely on the diffusion which takes place during the quiescent period in order to calculate fluxes. We have seen that (1) is a result of the flow, so may take it as given for the diffusion problem. We must also ensure that the initial condition set for the diffusion problem is at least approximately the same as the final condition for that part of the fluid which remains in the interface, since that serves as the initial condition for the next quiescent period.

This approach is a natural extension of that adopted successfully by Howard to describe thermal convection. One more element is necessary to allow a solution, namely a method for determining t_*. In Howard's treatment, diffusion proceeds until at time t_* the Rayleigh number evaluated across the thermal boundary layer exceeds a critical value. In the case of the diffusive interface the density distribution is determined by both components as shown in figure 4.
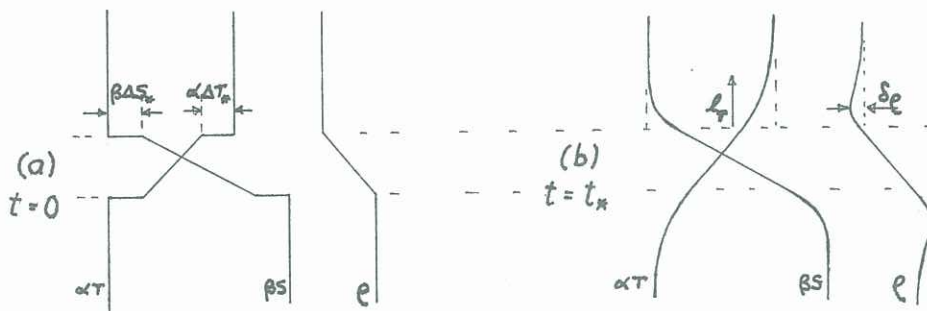

FIGURE 4   Profiles of S and T in the diffusion problem.

At least when $\tau \ll 1$, the obvious extension of Howard's approach is to evaluate the Rayleigh number in terms of the density difference $\delta\rho$ and length scale ℓ_T.

A full solution of the combined diffusion problem seems impracticable, and it is more fruitful to consider separately the simpler cases represented by the extremes of R_ρ and the extremes of

τ. These have not yet been worked through completely. However two points have emerged which are of interest. Firstly, although the function $F_T^*(R_\rho)$ predicted by these calculations agrees quite well with the power law (2) for both values of τ, it is not a power law, and does not match (2) at the extremes of R_ρ. The second point is an extension of the first, and relates to the value of F_T^* when R_ρ is large. Rather than F_T^* being small but finite at indefinitely large R_ρ, F_T^* exhibits a cut-off when $R_\rho = \tau^{-\frac{1}{2}}$ since

$$F_T^* \propto (1-R_\rho \tau^{\frac{1}{2}})^{4/3}$$

The reason for this cut-off is that at this R_ρ, ϵ and t_* both become infinite.

It seems, therefore, that we have a viable and self-consistent model of the diffusive interface. When it is properly worked through, it is likely to raise some questions about the experimental results for small and large R_ρ; for example the cut-off predicted by the model is supported by the experiments with $\tau = 0.02$, but not by those with $\tau = 0.33$. Nevertheless while such a model can be very helpful in illuminating the observed properties of the diffusive interface, it does not contain all the physics involved and must continue to be used with caution.

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