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A KARMAN-POHLHAUSEN TYPE APPROXIMATE METHOD FOR SOLVING SECOND ORDER BOUNDARY LAYER EQUATIONS

by

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SUMMARY

In this paper a Karman-Pohlhausen type method is developed for approximately solving the incompressible two-dimensional second order boundary layer equations, by expanding all the variables (momentum and displacement thicknesses, etc.) in an assymptotic series in powers of inverse square root of Reynolds number. To the first order we recover back the well known Karman-Pohlhausen method. Solutions are worked out for a few particular cases in longitudinal curvature problem corresponding to which similarity solutions exist. The results obtained thus not only compare well with the exact solutions but also are fairly insensitive to the order of polynomial approximation adopted for the velocity profiles. It is shown finally, but taking a simple example, that depending on its gradient even a positive curvature can lead to a positive increment to skin friction (in contrary to the usual notion).

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1. INTRODUCTION

It is well known that the second order boundary layer effects are important at lower Reynolds numbers which occur in situations like high altitude flights and high mach number wind tunnel tests. Estimation of these second order effects can be made exactly (apart from the numerical solutions, see reference (1), (2)) only for those situations which correspond to the similarity solutions (references (3), (4), (5)) in spite of the fact that the equations governing them are linear. A need therefore arises in general for a good approximate method for quick estimation of these effects.

Previous works of Devan and Oberai (6) and Werle and Davis (7), (8) using momentum integral equations have certain limitations as they had to make special assumption regarding the first order and second order boundary layer thicknesses (like whether one is greater than or equal to the other). In fact Werle and Davis (8) found, for the external vorticity problem, that the solutions were very sensitive to the above assumption. They also found that the results depend erratically on the order of polynomial approximation assumed for the velocity profile. They argued, in view of the linearity of the second order boundary layer equation, that equally poor results will be obtained for the curvature problem also, and finally concluded that the polynomial approach needs little further investigation.

In this paper we describe a method which is an extension of the Karman-Pohlhausen momentum integral method. It uses essentially a perturbation technique and does not have the limitations mentioned previously. The results are computed only for longitudinal curvature problem. They not only show good agreement with the exact solutions of Narasimha and Ojha (3), but are also fiarly insensitive to the order of polynomial assumed for the velocity profile, except when the boundary layer approaches separation.

2. MOMENTUM INTEGRAL EQUATION

Following Narasimha and Ojha (3) we define the total displacement and momentum thicknesses δ^* and θ in a simple way as

$$U_{os}\delta^* = \int_0^\infty \{U(Y\to 0) - u(y)\}\, dy = \delta_o^* + \varepsilon\delta_1{}^* + 0(\varepsilon^2) \;, \qquad (2.1)$$

$$U^2{}_{os}\theta = \int_0^\infty u(y)\{U(Y\to 0)u(y)\}dy = \theta_o + \varepsilon\theta_1 + 0(\varepsilon^2)$$

where all the variables are non-dimensionalized with a characteristic length scale L and characteristic velocity scale U_∞, U(Y) is the outer expansion [reference (3)] of the velocity along X, in the limit X,Y (coordinates along and normal to the surface) fixed and $\varepsilon \to 0$, defined by

$$U = U_o(X,\ Y) + \varepsilon U_1(X,\ Y) + 0(\varepsilon^2), \qquad (2.2)$$

u(y) is the inner expansion of the velocity in the limit $x = X$ and $y = Y/\varepsilon$ fixed and $\varepsilon \to 0$ defined by

$$u = u_o(x,\ y) + \varepsilon\ u_1(x,\ y) + 0(\varepsilon^2) \;, \qquad (2.3)$$

U_{os} is the surface velocity given by $U_o(y \to 0)$ and ε is the inverse square root of Reynolds number $U_\infty L/\nu$. The matching condition requires [reference (3)] that

$$u_o(y \to \infty) = U_o(Y \to 0), \qquad u_1(y \to \infty) = (\partial U_o/\partial Y)_{Y=0}\ y + U_1(Y \to 0). \qquad (2.4)$$

Momentum integral equations for the three problems, displacement thickness effect, longitudinal curvature and the external vorticity effects (to which the second order effects can be broken down because of the linearity of the second order boundary layer equations), have been derived in references (6), (7), and (9). Following reference (9) we write the combined momentum integral for both first and second order boundary layer equations as

$$U_{os}^2\ \theta' + U_{os}U_{os}'\ \theta(2 + f_1) = \tau + \varepsilon B \qquad (2.5)$$

where $\quad \tau = \tau_o + \varepsilon\tau\ , \quad f_1 = \delta_1{}^*/\theta, \qquad (2.5)a$

and B depends on one of the three second order effects depending on the effects for which the above equation (2.5) is used. The superscript dashes represent the derivatives with respect to the appropriate variable. We introduce the following variables defined by

$$Z = \theta^2, \quad \tau\theta/U_{os} = f_2, \quad \lambda = \delta^2 U'_{os}, \quad \alpha = \theta^2 U'_{os}, \quad F = 2\{f_2 - \alpha(2 + f_1)\}, \qquad (2.6)$$

where δ is the total boundary layer thickness. In these variables the momentum integral equation (2.5) reduces to the form

$$Z = F/U_{os} + 2\varepsilon Z^{1/2} B/U^2_{os}, \qquad \alpha = ZU'_{os}. \tag{2.7}$$

Now if we use the assymptotic expansion of the form

$$q = q_o + \varepsilon q_1 + O(\varepsilon^2) \tag{2.8}$$

for δ, λ, Z, α and F in (2.6) we get

$$Z_o = \theta^2_o, \quad \alpha_o = \theta^2 U'_{os}, \quad \lambda_o = \delta^2_o U'_{os}, \quad Z_1 = 2\theta_o\theta_1, \quad \alpha_1 = 2\theta_o\theta_1 U'_{os}, \quad \lambda_1 = 2\delta_o\delta_1 U'_{os}. \tag{2.9}$$

Use of (2.8) and (2.9) in (2.7) gives for the first order

$$Z'_o = F_o/U_{os}, \quad \alpha_o = Z_o U'_{os} \tag{2.10}$$

and for the second order

$$Z_1 = F_1/U_{os} + 2(Z_o)^{1/2} B/U^2_{os}, \quad \alpha_1 = Z_1 U'_{os}. \tag{2.11}$$

3. METHOD OF SOLUTION

The general procedure for solving (2.10) and (2.11) is as follows. Assume a polynomial in $\eta = y/\delta$ for the total velocity u. Evaluate the coefficients of this polynomial by satisfying as many boundary conditions as required. Given then the initial value of $\lambda = \lambda_o + \varepsilon\lambda_1$ which is the unknown parameter in the velocity profile, we can integrate (2.10) and (2.11) step by step. In the discussion what follows we restrict ourself to the longitudinal curvature problem. But the procedure laid out can be easily extended to the other effects also.

For the longitudinal curvature case $(\partial U_o/\partial y)$ at $y = 0$ in (2.4) and B in (2.5) are given by [reference (3), (10)]

$$(\partial U_o/\partial Y)_{Y=0} = -KU_{os}, \tag{3.1}$$

$$B = 2KU_{os} + K'(2I_1U_{os} + I_2) + KU_{os} I'_1, \tag{3.2}$$

where K is the non-dimensional longitudinal curvature (non-dimensionalized by length scale L) and I_1 and I_2 are the first moments of the first order mass and momentum defects given by

$$I_1 = \int_0^\infty (U_{os} - u_o) y\,dy, \quad I_2 = \int_0^\infty u_o(U_{os} - u_o)\, y\,dy. \tag{3.3}$$

3.1 *Polynomial approximation:*

We assume for the total velocity a profile given by a polynomial (say fourth order)

$$u/U_{os} = a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4. \tag{3.4}$$

where $\eta = y/\delta$. Equation (3.4) satisfies the boundary condition $u/U_{os} = 0$ at $\eta = 0$. The constants of the polynomials are evaluated by satisfying as many boundary conditions as required. We as usual assume that the boundary condition at $y \to \infty$ is satisfied at $\eta = 1$. The boundary conditions at $\eta = 1$ can be written from (3.1) and (2.4) as

$$u/U_{os} = 1 - K\delta, \quad \partial(u/U_{os})/\partial\eta = -K\delta, \quad \partial^2(u/U_{os})/\partial\eta^2 = 0, \quad \partial^3(u/U_{os})/\partial\eta^3 = 0, \text{ etc.} \tag{3.5}$$

We take an additional boundary condition at $\eta = 0$, obtained from satisfying the combined momentum equation at $y = 0$ which gives [reference (3)]

$$\partial^2 u/\partial y^2 + \varepsilon K\partial u/\partial y = \partial/\partial x \left(K\int_0^\infty (U^2_{os} - u^2_o)\, dy\right). \tag{3.6}$$

Using the definition of θ_o and η in (3.6) we get

$$\partial^2(u/U_{os})/\partial\eta^2 + \varepsilon K\delta\partial(u/U_{os})/\partial\eta = -\lambda + \varepsilon P_1, \tag{3.7}$$

where
$$P_1 = (\delta^2_o/U_{os})\, d/dx\, \{K\delta_o U^2_{os}(\delta^*_o + \theta_o)\}. \tag{3.8}$$

For the fourth order polynomial assumption we satisfy first three of (3.5) and (3.7) to evaluate a_1, a_2, a_3 and a_4, (for a third order polynomial first two of (3.5) and (3.7) and for a fifth order polynomial first four of (3.5) and (3.7)) which gives the velocity profile as

$$u/U_{os} = E(\eta) + \lambda G(\eta) + \varepsilon K\delta/p(E(\eta) + \lambda G(\eta) - pH(\eta)) + \varepsilon P_1 G(\eta) \tag{3.9}$$

where the functions E, G, H and the constant p are listed in table 1.

3.2 *Evaluation of the second order variables.*

Using (3.9) in the definitions of τ, θ, δ^*, α, F, I_1 and I_2 [see (2.5a), (2.6), (2.1) and (3.3)] we get

$$\tau\delta/U_{os} = h_1 - \varepsilon K\delta g_1/p, \quad \delta^*/\delta = h_2 - \varepsilon K\delta g_2/p, \quad \theta/\delta = h_3 - \varepsilon K\delta g_3/p, \quad I = U_{os}\delta_0^2 h_4 ,$$

$$I_2 = U_{os}\delta_0^2 h_5, \quad \alpha = \lambda(h_3 - 2\varepsilon K\delta h_3 g_3/p), \quad F = \Lambda_1 - \varepsilon K\delta\Lambda_2/p, \quad \Lambda_1 = 2h_3(h_1 - \lambda h_2 - 2\lambda h_3), \tag{3.10}$$

$$\Lambda_2 = 2h_3(g_1 - \lambda g_2 - 2\lambda g_3) + g_3/h_3\, \Lambda_1$$

where the functions h_1, h_2, etc. are all listed in table 1.

Using the assymptotic expansions given by (2.8) for τ, δ, δ^*, θ and λ and using a Taylor series expansion of all functions of λ around λ_0 we get

$$\tau_0\delta_0/U_{os} = h_1(\lambda_0), \quad \delta_0^*/\delta_0 = h_2(\lambda_0), \quad \theta_0/\delta_0 = h_3(\lambda_0),$$

$$\alpha_0 = \lambda_0 h_3^2(\lambda_0), \quad F_0 = \Lambda_1(\lambda_0),$$

$$(\tau_1\delta_0 + \tau_0\delta_1)/U_{os} = \lambda_1\, h_1'(\lambda_0) - K\delta_0 g_1(\lambda_0)/p,$$

$$\delta_1^* = \{\lambda_1 h_2'(\lambda_0) - K\delta_0 g_2(\lambda_0)/p\}\, \delta_0 + h_2(\lambda_0)\delta_1 ,$$

$$\theta_1 = \{\lambda_1 h_3'(\lambda_0) - K\delta_0 g_3(\lambda_0)/p\}\, \delta_0 + h_3(\lambda_0)\delta_1 ,$$

$$\alpha_1 = \lambda_1\{h_3^2(\lambda_0) + 2\, h_3'(\lambda_0)\} + 2K\delta_0\lambda_0 h_3(\lambda_0) g_3(\lambda_0)/p,$$

$$F_1 = \lambda_1\Lambda_1'(\lambda_0) - K\delta_0\Lambda_2(\lambda_0)/p. \tag{3.11}$$

Substitution of (3.9) in the expression (3.2) for B and the expression (3.8) for P_1 and, after simplification, we can put them in the form

$$2Z_0^{1/2}B/U_{os}^2 = (2K\delta_0/U_{os})\{W_1(\lambda_0) + W_2(\lambda_0)\lambda_0^2 U_{os}U_{os}''/U_{os}'^2 + W_3(\lambda_0)\lambda_0 K' U_{os}/KU_{os}'\},$$

$$P_1 = K\delta_0\{W_4(\lambda_0) + W_5(\lambda_0)\lambda_0^2 U_{os}U_{os}''/U_{os}'^2 + W_6(\lambda_0)\lambda_0 K' U_{os}/KU_{os}'\} \tag{3.12}$$

where W_1, W_2 . . . . W_6 are all given by

$$W_1 = 2h_3 + 2\lambda_0 h_3 h_4 + (h_4 + \lambda_0\, h_4')h_3\Lambda_1/(h_3^2 + 2\lambda_0 h_3 h_3'),$$

$$W_2 = (h_3 h_4' - 2h_3 h_4 h_3')/(h_3 + 2\lambda_0 h_3') ,$$

$$W_3 = h_3(2h_4 + h_5),$$

$$W_4 = 2\lambda_0(h_2 + h_3) + \Lambda_1(h_2 + h_3)/2h_3^2 + \lambda_0(h_2/h_3)'\, h_3\Lambda_1/(h_3^2 + 2\lambda_0 h_3 h_3'),$$

$$W_5 = (h_3 h_2' - h_2 h_3')/(h_3 + 2\lambda_0 h_3'),$$

$$W_6 = h_2 + h_3 . \tag{3.13}$$

We see from (3.12) that the parameters on which the second order boundary layer quantities depend are λ_0, $K\delta_0$, $U_{os}U_{os}''/U_{os}'^2$ and $K'U_{os}/KU_{os}'$: the first is the well known pressure gradient parameter, the second is the ratio of the boundary layer thickness to the radius of curvature, the third signifies the ratio of length scales associated with change in velocity and change in gradient of velocity and the fourth denotes the ratio of length scales associated with the change in velocity and change in curvature.

3.3 *Solution of momentum integral equations*

Method of solving (2.10) with $u/U_{os} = E(\eta_o) + \lambda_o G(\eta_o)$ reduces to the well known Karman-Pohlhausen method for solving first order boundary layer. The fact, that all the second order variables depend linearly on λ_1 [see equation (3.11)], can be used to reduce the momentum integral equation (2.11) to a linear first order differential equation which can be integrated easily. Substituting for Z_1, F_1 and B from (2.9), (3.11) and (3.12) and simplifying we can reduce (2.11) to

$$\phi' - \xi\phi = \psi_2' + \psi_1 \tag{3.14}$$

where

$$\phi = (\lambda_1/U_{os}')\, h_3(\lambda_o)\,[2\lambda_o h_3'(\lambda_o) + h_3(\lambda_o)]\,,$$

$$\xi = (U_{os}'/U_{os})\, \Lambda_1'(\lambda_o)/[h_3(\lambda_o)\{2\lambda_o h_3'(\lambda_o) + h_3(\lambda_o)\}]\,,$$

$$\psi_1 = -K\delta_o\Lambda_2(\lambda_o)/6U_{os} + (2/U_{os}^2)\, Z_o^{\frac{1}{2}}B\,,$$

$$\psi_2 = 2K\delta_o\lambda_o h_3(\lambda_o)\, g_3(\lambda_o)/p\,. \tag{3.15}$$

Equation (3.14) being a linear differential equation, solution can be written down easily as

$$\phi(x) = \psi_2(x) + \{\phi(0) - \psi_2(0)\}(\mathrm{Exp}\int_0^x \xi dx + \mathrm{Exp}\int_0^x \xi dx)\,[\int_0^x (\psi_1 + \xi\psi_2)(\mathrm{Exp}-\int_0^x \xi d\zeta)dx] \tag{3.16}$$

Given an initial value of λ_o and λ_1, $\phi(0)$ and $\psi_2(0)$ can be calculated and ϕx can then be obtained from (3.16) for the given $U(x)$, $K(x)$ and calculated $\lambda_o(x)$ from (2.10). If the boundary layer starts from a sharp edge the initial values of λ_o and λ_1 are zero as δ_o and δ_1 are zero and so $\phi(0) = \psi_2(0) = 0$. If the boundary layer starts from a stagnation point $\xi \to \infty$ at the stagnation point. If $\phi(x)$ has to be finite $\phi(0) = \psi_2(0)$ which gives a starting value of λ_1.

4. RESULTS AND DISCUSSIONS

We shall now consider, for evaluating the method, a class of similar solutions given by Narasimha and Ojha (3). For the class of similar solutions they considered first order and second order solutions were of the same form and both free stream velocity and curvature variations were given by power laws as

$$U_{os} = C_1 x^m, \text{ and } K = k(m+1)/2\; x^{\frac{1}{2}(m-1)} \tag{4.1}$$

where C_1 and k are constant. For these similar solutions λ_o is a constant (therefore F_o, α_o are also constants) and is related to m. This relation can be found by integration of (2.10) which on simplification gives

$$\alpha_o/F_o = U_{os}' \int dx/U_{os}\,. \tag{4.2}$$

Substitution for α_o, F_o in (4.2) from (3.11) and using (3.10) and (4.1) we get

$$2\{h_1(\lambda_o) - \lambda_o h_2(\lambda_o) - 2\lambda_o h_3(\lambda_o)\}/\{\lambda_o h_3(\lambda_o)\} = (1-m)/m \tag{4.3}$$

The Falkner-Skan pressure gradient parameter β is related to m by

$$m = \beta/2 - \beta \tag{4.4}$$

For a given β or a given m, λ_o can be obtained by using (4.4) and (4.3). Further for these similarity solutions the parameters $K\delta_o$, $U_{os}U_{os}''/U_{os}'^2$ and $K'U_{os}/KU_{os}'$ are also constants so that we can get an equation for λ_1, in a similar way as we obtained (4.3) for λ_o, by integrating (2.11) directly as

$$\alpha_1 = (F_1 + 2Z_o^{\frac{1}{2}}B/U_{os})\,U_{os}' \int (dx/U_{os})\,. \tag{4.5}$$

Substitution for α_1, F_1 and $2Z_o^{\frac{1}{2}}B/U_{os}$ from (3.11) and 3.12) and using (4.1) we get

$$\lambda_1\{h_3^2(\lambda_o) + 2h_3'(\lambda_o) - (m/m-1)\Lambda_1'(\lambda_o)\} = -K\delta_o[2\lambda_o h_3(\lambda_o)g_3(\lambda_o)/p + (m/m-1)\{\Lambda_2(\lambda_o)/p - 2W_1(\lambda_o)\}$$
$$- 2W_2(\lambda_o)\lambda_o^2 - W_3(\lambda_o)\lambda_o]\,. \tag{4.6}$$

Once λ_1 is obtained from (4.6) all second order quantities can be obtained from (3.11). A similar procedure can be adopted for polynomials of other orders. The results obtained thus for the three polynomials P_3, P_4 and P_5, (third, fourth and fifth order polynomials) are tabulated in Table 1 and 2. Table 1 contains the listing of functions h_1, h_2, h_3, etc. for the three polynomials considered. The results of increment to skin friction for various values of β in the range $1.00 > \beta > -0.1988$ (stagnation flow to separation) are given in Table 2 for the three polynomials. Exact solutions of Narasimha and Ojha are also given in Table 2 for comparison. The results of polynomial P_3 for $\beta = 1.0$ and 0.875 are not given as in these cases P_3 profile gives, inside the boundary layer, velocities higher than the free stream velocity. It is seen from Table 2 that $\Delta C_f (\Delta C_f = 2\tau_1)$ correction to skin friction coefficient agrees closely with the exact solutions, except when the boundary layer approaches separation ($\beta \to -0.198$). Further, the results are also fairly insensitive to the order of the polynomial again except near separation. This poor result near separation is not surprizing for we know that the separation point is a singular point of the second order boundary layer equations and integral methods are known to be not adequate near these singularities. Further, the first order boundary parameters themselves are not accurate near separation. Thus except possibly near the singularities of the second order boundary layer equations the above method can be used with confidence for estimating the second order parameters.

Next we consider a special simple case for which the pressure gradient in zero and curvature increases linearly with x as

$$K = C_1 x + C_2 \tag{4.7}$$

For this case $\lambda_o = 0$ and $\delta_o/\sqrt{\frac{x\nu}{U}} = 5.8356$ [reference (9)].

Evaluating δ_1 from (3.16) we get

$$\delta_1/\sqrt{\frac{x\nu}{U}} = 30.93\, C_1\, x^{3/2} + 25.58\, C_2. \tag{4.8}$$

Using (4.8) in (3.11) we get

$$\Delta C_f = -2(6.039 C_1 x + 3.355\, C_2) \tag{4.9}$$

If now $C_1 = 0$, curvature becomes constant and we get a negative ΔC_f for positive curvature (in accordance with the usual notion). We can easily see from (4.9) and (4.7) that by choosing C_1 and C_2 properly we can make ΔC_f to assume a positive value even when the curvature is positive.

Although we were concerned here with longitudinal curvature problem the procedure can be easily extended to the other two linear problems (displacement thickness and external vorticity effects) with little difficulty. We expect the general conclusions reached here to be valid for the other two problems because of the similar nature of the governing equations.

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Functions	ORDER OF POLYNOMIAL		
	P_3	P_4	P_5
p	4	6	8
E	$1/4\,(6\eta - 2\eta^3)$	$2\eta - 2\eta^3 + \eta^4$	$1/8\,(20\eta - 40\eta^3 + 40\eta^4 - 12\eta^5)$
G	$1/4\,(\eta - 2\eta^2 + \eta^3)$	$1/6\,(\eta - 3\eta^2 + 3\eta^3 - \eta^4)$	$1/8\,(\eta - 4\eta^2 + 6\eta^3 - 4\eta^4 + \eta^5)$
H	$-\eta - 3/4\,\eta^2 + 1/2\,\eta^3$	$-\eta - \eta^2 + 4/3\,\eta^3 - 1/2\,\eta^4$	$-\eta - 5/4\,\eta^2 + 5/2\,\eta^3 - 15/8\,\eta^4 + 1/2\,\eta^5$
h_1	$6/4 + \lambda/6$	$2 + \lambda/6$	$5/2 + \lambda/8$
h_2	$3/8 - \lambda/48$	$3/10 - \lambda/120$	$1/4 - \lambda/240$
h_3	$39/280 - \lambda/560 - \lambda^2/1680$	$37/315 - \lambda/945 - \lambda^2/9072$	$10/99 - \lambda/1584 - \lambda^2/31680$
h_4	$1/10 - \lambda_o/120$	$1/15 - \lambda_o/360$	$1/21 - \lambda_o/840$
h_5	$9/160 - 3\lambda_o/1120 - \lambda_o^2/4480$	$5/126 - \lambda_o/945 - \lambda_o^2/30240$	$325/11088 - \lambda_o/2016 - \lambda_o^2/126720$
g_1	$5/2 - \lambda/4 + P_1/K\delta$	$4 - \lambda/6 + P_1/K\delta$	$11/2 - \lambda/8 + P_1/K\delta$
g_2	$1/8 - \lambda/48 - P_1/12K\delta$	$1/10 + \lambda/120 - P_1/20K\delta$	$1/12 + \lambda/240 - 1/30 \quad P_1/K\delta$
g_3	$115/240 - 41/1680\lambda + \lambda^2/1680$ $-P_1/K\delta\left(1/140 + \lambda/210\right)$	$26/63 - 7\lambda/540 + \lambda^2/4536$ $-P_1/K\delta\left(2/215 + \lambda/756\right)$	$1091/2772 - 47\lambda/6160 + \lambda^2/15840$ $-P_1/K\delta\left(1/198 + \lambda/1980\right)$

Table 1. List of functions in equation (3.9) and (3.10) for the three polynomial assumption.

β	m	$\Delta C_f/kx^{(3m - \frac{1}{2})}$			
		P_3	P_4	P_5	Exact
1.0	1.0		-3.78	-3.87	-3.83
0.857	0.75		-3.46	-3.55	-3.50
0.667	0.5	-3.18	-3.18	-3.24	-3.17
0.4	0.25	-2.79	-2.83	-2.88	-2.78
0	0	-2.33	-2.24	-2.26	-2.05
-0.14	-0.0654	-2.09	-1.78	-1.66	-1.61
-0.195	-0.0888	-2.37	-1.34	-1.05	-2.12

Table 2. Comparison of the present results with the exact solutions.