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VARIATION OF THE TEMPERATURE FIELD IN ADIABATIC VISCOUS FLOW DUE TO DISSIPATIVE PROCESSES

by

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S U M M A R Y

Dissipative processes induce heat transfer and affect the temperature field within the adiabatic, viscous flow. Work, done by the forces, determined by the viscous stress tensor may be expressed as a sum of two terms. The first term is the work of displacement of the fluid element against friction forces. The second term is a dissipation function which is a measure of heat generated in the flow. The temperature field is a function of the work of friction and the paper presents the solution of this problem for laminar flow through cylindrical duct. The equation of energy to be solved is a parabolic equation with second order boundary conditions. The numerical solution as well as the analytical asymptotic one has been found.

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NOMENCLATURE

$c\,[J/kg\,K]$ - specific heat
$V\,[m/sec]$ - velocity
$\rho\,[kg/m^3]$ - density
$\lambda\,[J/m\,sec\,K]$ - heat conductivity
$\mu\,[kg/m\,sec]$ - viscosity
$u\,[J/kg]$ - internal energy

INTRODUCTION

The work, done by the forces, determined by the viscous stress tensor may be expressed (for a unit of mass and a unit of time) in the form

$$L = \vec{V}\cdot\vec{f} + \frac{1}{\rho}\,\phi$$

The first term on the right hand side is the work of displacement of the fluid elements against frictional forces. This term is of negative sign (the angle between $\vec{V}$ and $\vec{f}$ is obtuse) and this term alone causes the isentropic decrease of pressure and consequently decrease of temperature in a compressible flow. The second term (ϕ) is a dissipation function. It is the work of deformation of particles (*) and in the case of a compressible fluid also the work of change of volume (against friction forces) of particles. The dissipation function is always of a positive sign and it induces the increase of the entropy and temperature. This function is a measure of heat generated in the flowing fluid.

In this connection there is an essential difference between the compressible and incompressible flow. Apart from the effect of the change of the kinetic energy on the temperature, the temperature change caused by the dissipation function may be counterbalanced by the work ($\vec{V}\cdot\vec{f}$) in compressible flow. In the case of an incompressible flow the change of pressure does not affect the temperature; consequently, an infinite increase of temperature takes place along each stream line due to dissipation. The dissipation function is usually considered as a negligible value in a laminar flow. The relationship between the dissipation function and the temperature in the incompressible laminar flow through cylindrical tube is presented below.

In this case

$$\vec{V}\cdot\vec{f} = \frac{\mu}{\rho}\left(\frac{V}{r}\frac{dV}{dr} + V\frac{d^2V}{dr^2}\right) \qquad \text{and} \qquad \phi = \mu\left(\frac{dV}{dr}\right)^2$$

substituting the velocity profile $V = -\frac{1}{4\mu}\frac{dp}{dz}(R^2 - r^2) = const.(R^2 - r^2)$

into the sum of the above two expressions we obtain

$$L = \frac{1}{4\rho\mu}\left(\frac{dp}{dz}\right)^2(2r^2 - R^2) = const.\cdot(2r^2 - R^2)$$

The distribution of L and the distribution of both of its component along a radius (for a tube of radius R = 1) is illustrated in Fig. 1. It is evident from this diagram that $\vec{V}\cdot\vec{f} + \frac{1}{\rho}\phi = 0$ at the radius $r = \frac{1}{2}\sqrt{2}$ where the velocity takes its mean value.

The integral $\int_v dL$ over the region bounded by the walls of the duct and two arbitrary cross-sections is obviously equal to zero for any flow. The temperature field in a viscous adiabatic flow depends on the function L.

(*) Friction Force $\vec{f} = \frac{1}{\rho}\nabla\cdot\Pi = \frac{\mu}{\rho}\left[\nabla^2\vec{V} + \frac{1}{3}\nabla(\nabla\cdot\vec{V})\right]$

Mechanical energy equation $\frac{1}{\rho}\frac{dp}{dt} = -\frac{d}{dt}\left(\frac{V^2}{2}\right) + \vec{V}\cdot\vec{f}$

SOLUTION OF THE PROBLEM

From the energy equation

$$\frac{d}{dt}\left(u+\frac{p}{\rho}+\frac{V^2}{2}\right)=\frac{1}{\rho}\phi+\vec{V}\cdot\vec{f}+\frac{1}{\rho}\nabla\cdot(\lambda\nabla T) \tag{1}$$

and the relation (2)

$$\frac{1}{\rho}\frac{dp}{dt}=\vec{V}\cdot\vec{f}-\frac{d}{dt}\left(\frac{V^2}{2}\right)$$

we obtain for incompressible flow

$$\frac{du}{dt}=\frac{1}{\rho}\phi+\frac{1}{\rho}\nabla\cdot(\lambda\nabla T) \tag{3}$$

For the flow under consideration $V=V_z(r)$, $\phi=\phi(r)$ the specific heat C, heat conductivity λ, and viscosity μ are assumed to be constant. Taking into account the above assumptions and the relation

$\frac{d(\,)}{dt}=\vec{V}\cdot\nabla(\,)$ the equation (3) may be transformed to the form

$$V\frac{\partial T}{\partial z}=\frac{1}{\rho c}\phi+\frac{\lambda}{\rho c}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right) \tag{4}$$

Using the following substitutions:

R_0 - radius of the tube

$$A_1=\frac{R_0^2}{4\mu}\left(-\frac{dp}{dz}\right)=V_{max} \quad \text{(at the axis)}$$

$$A_2=\frac{4\mu}{\rho c R_0^2}A_1^2=\frac{R_0^2}{4\mu\rho c}\left(\frac{dp}{dz}\right)^2;\quad A_3=\frac{\lambda}{\rho c R_0^2}$$

we obtain

$$A_1(1-r^2)\frac{\partial T}{\partial z}=A_2 r^2+A_3\frac{\partial^2 T}{\partial r^2}+\frac{A_3}{r}\frac{\partial T}{\partial r} \tag{5}$$

with the boundary conditions

$$\frac{\partial T}{\partial r}=0 \quad \text{for } r=0 \text{ and } r=1;\quad T(r,0)=0$$

In equation (5) and in the following work r denotes a non-dimensional quantity in the interval $\langle 0,1\rangle$. Equation (5) together with the boundary conditions is the final formulation of the problem to be solved. This equation can be transformed by means of a Laplace transform (with respect to z) to the ordinary equation

$$A_1(1-r^2)sf=A_2r^2\frac{1}{s}+A_3\frac{\partial^2 f}{\partial r^2}+A_3\frac{1}{r}\frac{\partial f}{\partial r} \tag{6}$$

where $f=f(s,r)$; with the boundary conditions

$$\frac{\partial f}{\partial r}=0 \quad \text{for } r=0 \text{ and } r=1.$$

The explicit solution of equation (6) has not yet been found. An approximate solution may be obtained by assuming an expression for its first derivative in the form

$$\frac{\partial f}{\partial r}=a_1(r-r^2)+a_2(r-r^3)$$

which satisfies the boundary conditions $\frac{\partial f}{\partial r}=0$ for $r=0$ and $r=1$.

Hence

$$f=a_1\left(\frac{1}{2}r^2-\frac{1}{3}r^3\right)+a_2\left(\frac{1}{2}r^2-\frac{1}{4}r^4\right)+a_3 \tag{7}$$

coefficients a_1, a_2, a_3, have been found from the condition that expression (7) satisfies equation (6) at three points r = 0, ½, 1.

Applying the inverse Laplace transformation to expression (7) we obtain

$$T = \frac{7}{128}\frac{A_2}{A_3}e^{-\alpha z} + \frac{A_2}{A_1}z - \frac{7}{128}\frac{A_2}{A_3} - \frac{7}{22}\frac{A_2}{A_3}e^{-\alpha z}\left(3r^2 - 2r^3\right) + \left(\frac{21}{22}\frac{A_2}{A_3}e^{-\alpha z} + \frac{1}{2}\frac{A_2}{A_3}\right)\left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right) \tag{9}$$

where $\alpha = 34{\cdot}91\,\frac{A_2}{A_1}$

If $z \to \infty$ expression (9) takes the form

$$T = 0{\cdot}125\,\frac{A_2}{A_3}\left(2r^2 - r^4\right) + \frac{A_2}{A_1}z - \frac{7}{128}\frac{A_2}{A_3} \tag{10}$$

and this is an exact asymptotic solution of the original problem, satisfying equation (5) and the boundary conditions for r = 0 and r = 1.

Although the boundary condition $T(0,r) = 0$ has been included in the transformation of the form (5) into (6), the expression (9) satisfies this condition only at particular points (r = 0 and r = ½); however, this does not affect the asymptotic solution.

From equation (10) we obtain

$$\frac{\partial T}{\partial z} = \frac{A_2}{A_1} = \frac{1}{\rho c}\left(-\frac{dp}{dz}\right) \quad = \text{const.} \tag{11}$$

For the mean value of temperature over the cross-section of the duct (T_m), the formula (11) can be derived directly from the equation of energy. Formula (11) is valid for laminar as well as for turbulent flow and for an arbitrary shape of the cross-section of the duct, if $V(r)$ does not change along z. The gradient of the mean temperature T_m, as well as the component of the grad T along each particular stream line for large z, may be expressed for laminar flow as follows

$$\nabla T = \frac{A_2}{A_1} = \frac{1}{\rho c}\left(-\frac{dp}{dz}\right) = \frac{1}{\rho c}\frac{4\mu V_{max}}{R_0^2} = \frac{1}{\rho c}\frac{8\mu V_{max}}{R_0}$$

where V_m - mean velocity, R_0 - radius of the tube

Equation (5) has also been solved by a numerical method. The numerical integration has been performed using the ordinary finite difference method. Making the substitutions.

$$\frac{\partial T}{\partial z} = \frac{T_{i,k+1} - T_{i,k}}{l};$$

$$\frac{\partial T}{\partial r} = \frac{T_{i+1,k} - T_{i-1,k}}{2h}; \quad \frac{\partial^2 T}{\partial r^2} = \frac{T_{i+1,k} - 2T_{i,k} + T_{i-1,k}}{h^2}$$

where $r_{i+1} = r_i + h;\ h = 0{\cdot}1;\ z_k = kl;\ l = nh$

we obtain the following form of difference equation

$$T_{i,k+1} = T_{i,k} + \frac{A_2}{A_1}nh\frac{r^2}{1-r^2} + \frac{A_3}{A_1}\frac{h}{1-r^2}\left[\frac{1}{h}\left(T_{i+1,k} - 2T_{i,k} + T_{i-1,k}\right) + \frac{1}{2r}\left(T_{i+1,k} - T_{i-1,k}\right)\right] \tag{13}$$

which corresponds to the differential equation (5). In order to satisfy the boundary condition $\frac{\partial T}{\partial r} = 0$ for r = 0 and r = 1, the mesh has been chosen so that the boundaries defined by r = 0 and r = 1 lie halfway between the mesh lines, and it has been assumed that $T_{0,k} = T_{1,k}$ and $T_{10,k} = T_{11,k}$. [redacted]. Two examples of the evaluation of the temperature field are presented below for two kinds of liquid, both flowing through a tube of radius

EXAMPLE 1.

The evaluation has been performed for water at $T = 273\,K = 0°C$

$V_{max} = 10;\ \rho = 10^3;\ c = 4239;\ \lambda = 0{\cdot}552;\ \mu = 1{\cdot}79 \times 10^{-3}$

The increase of the mean temperature

$$\frac{dT_m}{dz} = 0.1683 \times 10^{-3} \,°K/m$$

The numerical (formula 13) and analytical (formula 10) solutions are produced in Table 1.
The small differences between T_{num} and T_{an} can be made even smaller, if instead of constant term equal $\frac{7}{128} \frac{A3}{A2}$ in formula 10, the corrected constant is found from relation $\int_F T V \, dF = 0$ for $z = 0$ (F - area of the cross-section of the duct)

which corresponds to the initial condition $T_m = 0$ for $z = 0$.
In example 1, $\frac{7}{128} \frac{A_2}{A_3} = 0.071$ and the corrected const. = 0.081. Table 2 presents the numerical and analytical results, where the latter were obtained using the corrected constant.

EXAMPLE 2

Properties of the liquid (glycerine)

$$\rho = 900; \quad c = 1950; \quad \lambda = 0.128; \quad \mu = 21.6 \times 10^{-3}; \quad V_{max} = 15$$

The increase of the mean temperature

$$\frac{dT_m}{dz} = 7.38 \times 10^{-3} \,K/m$$

Table 3 presents temperatures determined by the numerical and analytical method, with the use of the corrected value of the constant in the latter. For k = 400 ($z = 2 \times 10^3$) the differences are very small except in the vicinity of the axis of the duct. Both methods give almost identical results from k = 600 ($z = 3 \times 10^3$)

CONCLUSIONS

Although the increase of the temperature along z is infinite, the function $\frac{\partial T}{\partial r}$ remains constant with respect to z, after a certain initial distance from the entry to the tube. The increase of the temperature along the radius is proportional to the ratio μ/λ and V^2_{max}.
The temperature field does not depend on the Prandtl number.

Both examples verify the validity of the analytical method of solution of the problem, and this method can be applied to any similar problem of heat transfer with the same type of second order boundary conditions.

There are limitations in this method (for a viscous fluid flow problem) arising from the initial assumption μ = constant. The results of example 2 are only approximate ones, because the temperature differences even along the radius are relatively large, and this results in a significant variation of μ.

But for V_{max} = 5 instead of 15, the maximum difference in the temperature between r = 0 and r = 1 is $2\,°K$ instead of $18\,°K$ and difference in the temperature between $k=0$ $(z=0)$ and $k=600$ $(z=3000)$ is $8.4\,°K$. For such a temperature gradients the assumption μ = constant is still permissible.

The paper presents the quantitative assessment of the influence of the dissipation function on the temperature field in a viscous flow.

Table 1

	1	3	5	7	9	10
	0.05	0.25	0.45	0.65	0.85	0.95
	0.09	0.11	0.15	0.19	0.23	0.24
	0.099	0.118	0.157	0.206	0.248	0.259

Table 2

	0.05	0.25	0.45	0.65	0.85	0.95	
	0.09	0.11	0.15	0.19	0.23	0.24	K=200
	0.089	0.108	0.147	0.196	0.238	0.249	

	0.05	0.25	0.45	0.65	0.85	0.95	
	0.25	0.27	0.31	0.36	0.40	0.41	K=400
	0.258	0.277	0.316	0.366	0.407	0.418	

Table 3

	0.05	0.25	0.45	0.65	0.85	0.95	
	6.25	8.09	12.11	17.92	22.12	23.90	K=400
	5.363	7.568	12.18	17.925	22.818	23.983	
	12.83	14.92	19.35	24.96	29.78	31.07	K=600
	12.747	14.952	19.564	25.309	30.202	31.367	

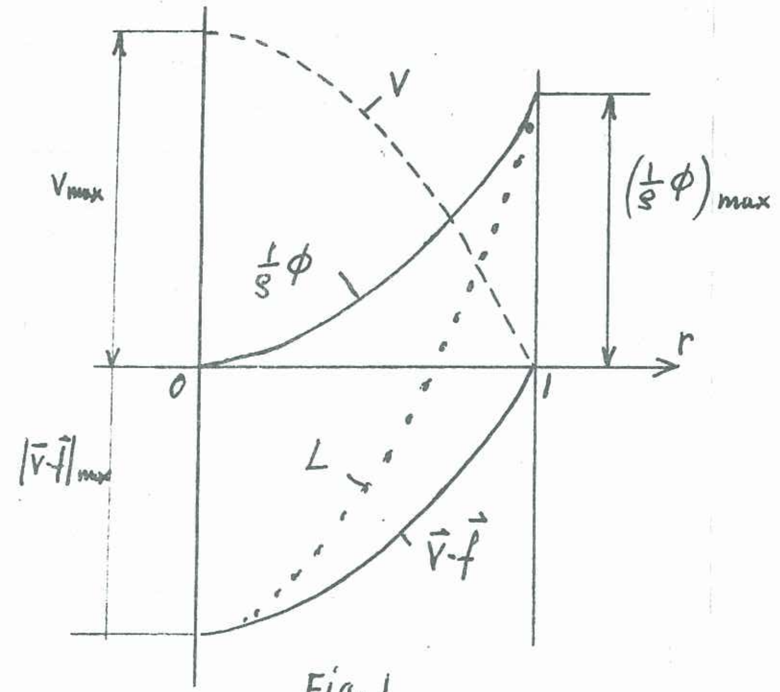
V
Vmax
$\left(\frac{1}{s}\phi\right)_{max}$
$\frac{1}{s}\phi$
r
0
1
$|\vec{V}-\vec{f}|_{max}$
L
$\vec{V}-\vec{f}$

Fig. 1