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DESCRIPTION OF WAKES BY VORTEX SHEETS

by

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SUMMARY

The use of vortex sheets to describe boundary layers and wakes is discussed and it is shown that two is the least number of sheets which will allow both displacement thickness and momentum thickness to be modelled. A wake can be described by four sheets (two from each boundary layer which forms it). The strength of each sheet is arbitrarily chosen as half the total vorticity, giving a velocity discontinuity of half the total velocity deficiency of a wake. Some comparisons are made between measured and modelled flows around an inlet guide vane of an axial flow compressor.

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NOTATION

C_D	drag coefficient of a blade in a cascade
C_L	lift coefficient of a blade in a cascade
c	blade chord
H	shape factor for boundary layer or one side of a wake
h_1,h_2	distance of vortex sheets from middle of wake
s	blade spacing
U	free stream velocity
u_m	lowest velocity in wake
$u' = (U-u_m)/U$	dimensionless velocity deficiency of wake
v_m	meridional (i.e. axial) component of mean velocity
v_u	peripheral component
α_m	angle of vector mean velocity through a blade row
δ_1	displacement thickness
δ_2	momentum thickness
Γ	total circulation around blade.

Introduction

The idea of using vortex sheets to replace the wake of an aerofoil originated with Lanchester (1). This was the simplest model, having a single vortex line to replace the aerofoil and turning it near the tips to replace the tip vortices. No quantitative analysis was attached to this model by Lanchester and little attention was paid to it at the time.

A few years later (1912) Joukowsky (2) had similar ideas. He had been impressed by some photographs made by Flamm of the wake of an isolated propeller in water. Many such photographs are now available and all give the clear impression of a region affected by the blade (the wake) enclosed within another region of inviscid flow (potential flow). Photographs of water flow downstream of a propeller often have the viscous wakes enhanced by the presence of cavitation bubbles which, together with the presence of turbulent eddies, clearly define them.

Joukowsky presented the idea of vortex sheets as discontinuities in the velocity field, and seems to have been the first to use the term "bound vortices" to describe vortex sheets replacing the propeller blade. It was he who was responsible for many of the basic ideas of the action of moving vortex sheets as a model to describe blade wakes.

He worked on these ideas for several years (2) but eventually decided that the difficulties of describing them by analytical methods was too great to justify further study. He abandoned the idea of discontinuities and replaced it with a model in which the vorticity was smoothed over the whole circumferential space downstream of the propeller. The whole field of the flow downstream of the propeller is then modelled as a rotational flow, which is a much less realistic model than that of a viscous wake embedded in a potential flow however one describes the viscous wake.

Unfortunately the second model proposed by Joukowsky has been the one most widely studied. It has led to a group of methods of prediction of flow through axial flow compressors which may loosely be described as streamline curvature methods, so called because they allow a correction to the simple two-dimensional equations for rotational flow to take into account the curvature of an average streamline. These methods have been widely touted and have achieved some success when properly handled under a limited range of conditions.

The big step forward in the development of vortex sheet models of boundary layers and wakes came in 1918 when Prandtl introduced his ideas (3) of a continuous vortex sheet replacing the wake. This vortex sheet was described by elementary vortex lines termed "horseshoe vortices". Prandtl was apparently thinking in terms of the barred horseshoes used on the cobblestones of German cities at that time. He was obviously fully aware of the "starting vortex" generated at the trailing edge of an aerofoil as it started to produce some lift. This model was called "lifting line theory" but the name led to so many misconceptions that it has more recently fallen into disrepute.

Prandtl's theory was very successful in predicting the induced drag,that is the effect of the streamwise components of the vorticity in the wake. A big advantage was that it used potential flow methods and these equations are linear so solutions can be added to give other solutions.

Unfortunately, some authors have thought of the induced drag as being a potential flow effect because potential flow methods are used to calculate it. A more realistic concept is that the whole effect of the viscosity has been confined to a single discontinuity, the vortex sheet itself. This is the thin boundary layer and wake approximation, but it should not be forgotten that, in subsonic flow, vorticity is confined to regions affected by viscosity, that is the whole of the vorticity is confined to the boundary layer and wake.

More recently, Lighthill (4) has produced considerable clarification to ideas about boundary layers by using the concept of vorticity.

He showed that the boundary layer simplification in two dimensional flow is equivalent to stating that the sum of all the vorticity at any section at any instant in time is equal to the free stream velocity U. He also showed that the centroid of all this vorticity is at a distance from the wall equal to the displacement thickness. Furthermore, the average velocity of convection of this vorticity is half the free stream velocity. He also made some extremely useful statements about the amount of streamwise vorticity induced by curvature of the streamlines of the potential flow outside the viscous regions in three dimensional flow.

The usual definition of the Biot-Savart law is based on electro-magnetic theory in which context it was derived but Lighthill's ideas allow an intuitive definition in the context of fluid flow.

It seems simplest to start from the picture of some body with a well developed boundary layer in a steady flow. If one introduces some disturbance to the flow, such as moving a Pitot tube towards the wall, then the velocity near the wall is altered. In other words the movement of the Pitot tube creates new vorticity at some places, and destroys it at others by altering the sum of the vorticity which is the free stream velocity.

The creation of new vorticity is thus associated with an increase in the free stream velocity and a consequent change in pressure. This creates a pressure wave which travels with the velocity of sound through the moving fluid and alters pressures and velocities at other points in the flow.

When the flow is incompressible (local velocities very much less than the velocity of sound) the pressure wave travels as a spherical wave from each point where the flow was disturbed and the velocities induced by the new vorticity are obtained from the Biot-Savart law.

The pressure wave at higher Mach Number is no longer spherical because, not only does the local fluid velocity alter from point to point, but so also does the local velocity of sound which is the local velocity of the pressure wave. One has thus, at least conceptually, an equivalent of the Biot-Savart law for compressible fluid flow. Unfortunately the complexities of compressible fluid flow are such that it is not yet possible to take advantage of the more general description of induced velocity.

Probably the most important advantage of a model based on vorticity concepts is that it is a purely kinematic description of the flow. This means that a vortex sheet model describes the geometry of the flow without asking how this particular geometrical pattern has arisen. One need not ask what pressure distribution was required until the geometry has been described. The same statement is valid in unsteady flow; a flow pattern can be described by a set of vortex sheets even though these sheets change position and strength as time changes.

The only restriction is that proposed by Helmholtz, which is that vorticity can be created or destroyed only at solid boundaries. Furthermore, vorticity is a property of fluid particles and is convected with them except for the diffusion by viscosity. This latter statement is not a restriction but part of the mechanism whereby a description of the vorticity distribution described the flow. When a turbulent viscous region is modelled by vortex sheets it is often convenient to lump the convection effects of the local turbulence into a local eddy viscosity but this is not always valid. However the concept of eddy viscosity is not necessary to describe a flow by vortex sheets. It only becomes necessary when one asks the question as to how this flow got there.

Two Vortex Sheets to Describe a Boundary Layer

The author of this paper was led to the idea of modelling the viscous regions by vortex sheets by a long term study of the details of the flow in an axial flow compressor and it is in this context that the remainder of this paper is presented.

The examination of the flow through the rotor of a fan or one stage of a compressor naturally leads to a study of the momentum changes. The simplest treatment leads to the well known Euler equation.

When one starts to add the effects of drag by studying the progress of wakes of blades through a successive blade row it immediately becomes apparent that,when the boundary layer on the blade is replaced by a single vortex sheet at a distance equal to the displacement thickness from the wall, the momentum deficiency of the boundary layer is neglected.

The author has shown elsewhere (5) how the momentum deficiency can be described by two vortex sheets. The conclusions are:

(1) it is necessary to use at least two vortex sheets to replace a boundary layer when one wishes to model the momentum deficiency as well as the mass flux deficiency.

(2) two vortex sheets require three parameters to determine their strength and position whereas only the two parameters of displacement and momentum thickness are required in this examination so that there is a certain arbitrariness in their definition.

(3) two vortex sheets, each of strength equal to half the total vorticity in the boundary layer at that section, placed so that their mean distance from the wall is equal to the displacement thickness and the distance between them is four times the momentum thickness, describe a potential flow which has the same mass flux deficiency and the same momentum flux deficiency as the viscous region and are sufficient to describe the latter for many practical purposes.

A vortex sheet may also be considered as a discontinuity in velocity so that replacing a viscous boundary layer by two vortex sheets is the same thing as replacing the actual curve of the velocity distribution by two rectangles. The results of this action for laminar boundary layers in two different cases are shown in Figures 1 and 2.

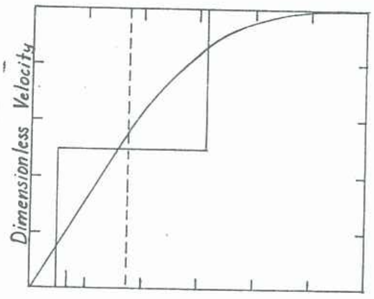

Fig. 1. Zero Pressure Gradient

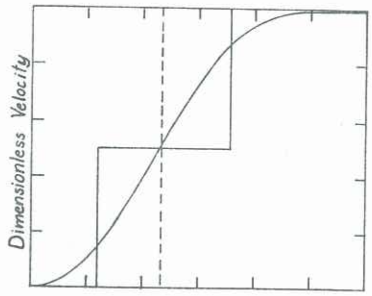

Fig. 2. Zero Wall Shear Stress

A big advantage of using two vortex sheets is that one is no longer restricted to thin boundary layers, or to situations where there is small velocity gradient in the direction normal to the wall in the potential flow outside the boundary layer.

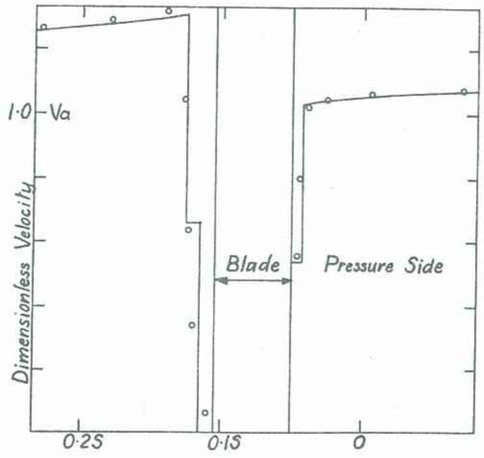

Fig. 3. Boundary Layers on Inlet Guide Blade of Fan

Figure 3 shows velocities measured near the inlet guide blades of an axial flow fan (Vortex Wind Tunnel). The traverse has been made in the circumferential direction at a station at one sixth of the chord from the trailing edge. The boundary layer on the pressure side was laminar and that on the suction side in transition from laminar to turbulent flow and separated. The Reynolds Number based on vector mean velocity and blade chord was about 100,000. The space/chord ratio was nearly 1.0, and the blade section is the British C4 shape. The potential flow has been calculated by a method of singularities following Martensen (6) with some modifications along the lines proposed by Hess and Smith (7).

The blade circulation connected with the lift has produced velocity gradients in the potential flow outside the boundary layers which render the usual definitions of displacement thickness and momentum thickness invalid for calculating their values from the measurements.

Vortex sheets have been developed by a method described in the next section of this paper and the blade replaced by them. The velocity distribution calculated by potential flow methods using the vortex sheets is shown in Figure 3. It is clear that there is now a potential flow solution which can be made to replace the boundary layer by a flow with the same mass flux and the same momentum flux.

No great pains have been taken to get very close agreement between these properties for the two velocity distributions, partly because there are severe doubts about the velocity measurements near the blade. Other evidence suggests that the boundary layer on the suction side was separated and equipment was not available at the time of writing to enable hot wire anemometer measurements to give good values across a separated region. Some of the points plotted near the wall may have the wrong sign, or at the very least be corrupted by the presence of the large eddies associated with transition. Another reason for not spending more time on this detail is that there are other more important difficulties yet to be overcome at the trailing edge itself. Agreement between the predicted and measured velocity distribution is accepted as satisfactory at this stage.

Vortex Sheets Describing a Wake

Far downstream of the trailing edge of an aerofoil the effects of circulation have died out and the usual definitions of displacement thickness can be used to calculate their values from measurements. The author has previously shown (5) how two vortex sheets can then be used to replace the wake in a manner similar to that outlined above for boundary layers. It has since been found more satisfactory to use four sheets, two from the boundary layers of each surface of the blade.

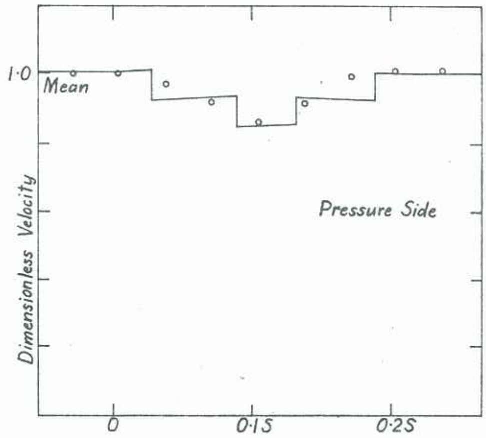

Fig. 4. Velocity Distribution in Wake.

Figure 4 shows measured and predicted velocity distribution through the wake of the same blade as that used for Figure 3. The station was at one third of the blade chord downstream of the trailing edge.

The vortex sheets have been chosen so that each has a strength equal to half the velocity deficiency of the wake.

The equations of conservation of mass and momentum require for each half of the wake that

$$U\delta_1=\frac{1}{2}(U-u_m)(h_2+h_1) \qquad (1)$$

and

$$U^2\delta_2=(U-u_m)U_m h_1+\frac{1}{2}(U-u_m).\frac{1}{2}(U+u_m)(h_2-h_1) \qquad (2)$$

From these, expressions for h_1 and h_2 are readily obtained in the non-dimensional forms

$$h_1/\delta_2=[(1-\frac{1}{2}u^1)H-1]/\frac{1}{2}(u^1)^2 \qquad (3)$$

$$h_2/\delta_2=(H-\frac{1}{2}u^1/\delta_2)/\frac{1}{2}u^1 \qquad (4)$$

Equations (3) and (4) are used to calculate the positions of the sheets from predictions of wake decay.

The situation of Figure 4 appears to allow calculation of δ_1 and δ_2 by the usual methods but a potential flow calculation of the velocities at this station due to the blade circulation around the "bare blade" (i.e. without any simulated boundary layer or wake) showed that there was still an apparent velocity defect of about 1% near the middle of the wake. The difference between the momentum thickness calculated in the usual way and that calculated from the difference between the measured velocities and the bare blade potential flow values was about 10%. Calculation of the required thicknesses from the position and strength of the vortex sheets which model the measured wake is thus much more accurate than the usual method.

The vortex sheet model can also be used to describe the viscous region in the vicinity of the blade trailing edge where the effect of circulation is more pronounced as shown in Figure 5.

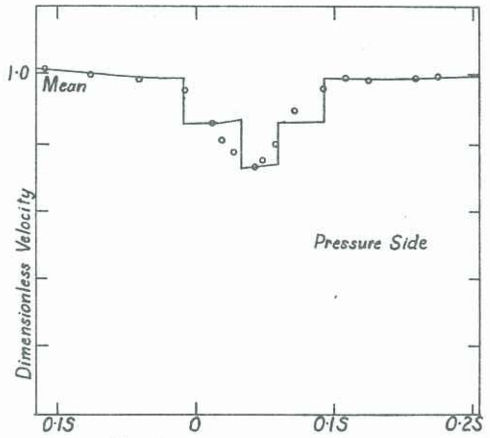

Fig. 5. Velocity Distribution in Wake

Measurements for Figure 5 were made at a station half way between that of Figure 4 and the trailing edge or at one sixth of the chord downstream of the trailing edge. At this station the method of prediction has placed the middle of the wake too far to one side of the measured position but the relative position and strengths of the four sheets is considered satisfactory at this stage of the development of the model.

The Trailing Edge Problem

It is still not clear as to what should be done to best describe the situation in the near vicinity of the trailing edge with vortex sheets. This appears to be a difficult problem and reliable data is hard to obtain.

Any practical fan blade has a rounded trailing edge. The boundary layer from each surface separates and later joins the layer from the other surface to form a separation bubble with two recirculating regions of different strength.

The boundary layer on the suction surface develops on a longer adverse pressure gradient than that on the pressure surface and, at the design point, is usually about twice as thick as the other. It is usually fully turbulent and accounts for up to three quarters of the friction loss.

When the boundary layers unite to close the trailing edge separation bubble there is considerable rapid mixing in the early stages. The mixing eddies are large and strong, the boundaries of the separation bubble fluctuate widely and the momentum deficiency increases almost as rapidly as it does on the aerofoil itself.

The greater part of the mixing loss seems to occur in the first three or four wake displacement thickness but it goes on increasing for some distance. As the wake decays it keeps on spreading and taking energy from the free stream amounting to perhaps 20% of the total drag by the time the wake has reached a point one chord length from the trailing edge. This effect is usually called the mixing loss.

Far downstream of the blade row where the wake is widely diffused the well known simple momentum balance gives

$$C_L + C_D \tan \alpha_m = 2(s/c)\, \Delta v_u/v_m = 2\Gamma/v_m c \qquad (5)$$

The same expression can be shown to apply closer to the blade. This means that the circulation around any circuit increases as the point at which the circuit crosses the wake is moved downstream. It also seems likely that there is a net shear force on a section which crosses the wake. The net result of all this is that the wake is initially curved to accommodate the initial momentum deficiency then gradually straightens out as the wake decays.

The method of treating the problem at the time of writing is to continue the vortex sheets past the blade trailing edge to give them positions and strengths which appear reasonable. A result of this arbitrary treatment is shown in the velocity distribution of Figure 6.

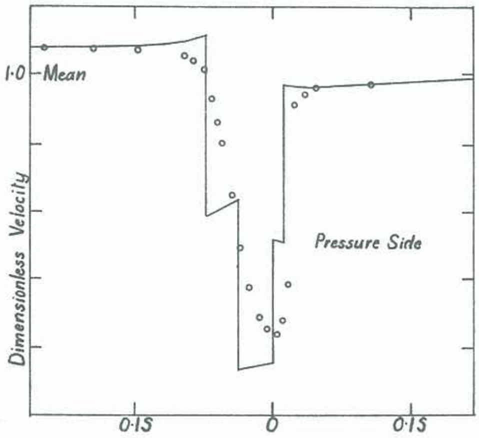

Fig. 6. Velocity Distribution.

The measuring station to which Figure 6 refers is nearly 2% of the chord downstream of the trailing edge of the blade referred to in the previous figures.

Several points of error in the predicted wake are obvious:

(1) the strength of the vortex sheets on the pressure side is too high. This is because there is no allowance for curvature of the sheets in calculating their strength.

(2) the whole predicted wake should be shifted towards the pressure surface side.

(3) the peak in velocity on the pressure side is incorrect.

Most of these errors in prediction can be considerably decreased by altering the curvature of the sheets in this vicinity, but at present this must be done arbitrarily and no new light is shed on the real behaviour.

The most important deficiency in the system remains as the lack of a suitable replacement for the Kutta-Joukowsky trailing edge hypothesis. The extremity of the trailing edge separation bubble now replaces the blade trailing edge but there must be another requirement additional to the one that the flow must leave the blade trailing edge smoothly.

Summary and Conclusions

(1) Two vortex sheets are required to model the mass flux and momentum flux deficiencies of a two dimensional boundary layer flow.

(2) The two vortex sheet model is not restricted to thin boundary layers in nearly plane flow but is equally applicable to thick boundary layers, curved and separated flow. Values of

displacement and momentum thicknesses are sometimes more accurately calculated from the positions and strengths of the sheets which model the flow than by conventional methods, provided the potential flow outside the boundary layer is well described by the calculations from the model.

(3) New and more detailed questions about the behaviour of the flow near the trailing edge of a lifting aerofoil have been raised without any satisfactory answers being given. (The measurements reported here are of little use in this regard because they have been made in a fan where there are strong cross flows near the trailing edge and these convect vorticity from other radii in to the test section).

(4) Away from the trailing edge, the boundary layers and wakes of the inlet guide blades of an axial flow compressor have been described in a manner adequate for many purposes by a potential flow calculation for which the only data has been the mean axial velocity, the blade angles, the published co-ordinates of the symmetrical basic section, the blade chord and the space/chord ratio

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