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A CLASS OF ASYMPTOTIC SOLUTIONS IN THE THEORY OF NON-LINEAR THERMAL CONVECTION

by

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SUMMARY

In this paper asymptotic solutions of the mean field equations for large values of the Rayleigh number and horizontal wave number have been considered. Appropriate analytical and numerical solutions have been derived and it is found that the character of these solutions differs quite markedly from the asymptotic solutions obtained when the horizontal wave number is of order one.

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1. INTRODUCTION

Solutions of the equations governing convective heat transfer in a horizontal layer of fluid heated from below can be placed in one of several categories according to a broad wave number classification. More precisely, the range of wave numbers, a, that support convection are determined by the condition $\gamma \leqslant 1$, where

$$\gamma = \frac{(a^2+\pi^2)^3}{a^2 R} = \frac{R_0}{R} , \tag{1}$$

R is the Rayleigh number and R_0 the critical Rayleigh number for the same a. The horizontal extent of the convective cells is proportional to 1/a and in the case of free boundaries the calculated value of a_c, for the onset of convection, is $\pi/\sqrt{2}$ which is in accord with experimental results.

The wave numbers of normal physical interest are those that maximize the total convective heat transport across the fluid layer and the calculation of these values has been the objective of many non-linear theoretical studies and experimental investigations.

Asymptotic solutions of the mean field equations for large values of the Rayleigh number have also been considered by a number of authors (Howard 1965 [1]; Herring 1966 [2]; Roberts 1966 [3]; Van der Borght 1971 [4]) and these solutions have recently been extended to take into account the effects of a magnetic field and rotation (Van der Borght, Murphy and Spiegel 1972 [5]; Van der Borght and Murphy 1973 [6]; Van der Borght and Murphy 1973 [7]). All these solutions made the assumption that the horizontal wave number a was of order one. The principal purpose of this paper is to examine the case when both a and R are large, to show that the character of the solutions changes markedly for these values of the parameters and to derive the appropriate analytical and numerical solutions.

It is also convenient to refer to the vertical convective velocity distribution W(z), where the extent of the layer is $0 \leqslant z \leqslant 1$, as a means of identifying the class of solutions considered in this paper. When $\gamma \simeq 1$ and $a \ll \pi$, corresponding to a value of the Rayleigh number just greater than the critical value R_0 for the onset of convection, weak convection results and the W(z) profile is nearly a sine curve and the solutions of the problem are well represented by the first two terms of the expansions (33). For values of the wave number $a \simeq \pi$ the W(z) solution still has a sine type profile but the numerical solution of the basic equations requires many more terms in the representation (33). For example, when $R = 10^7$ strong convective motions exist and to ensure satisfactory convergence of the numerical technique ninety terms were included (i.e. M = 90 [5]). Notwithstanding the large number of terms required for the numerical solutions at large R when $a \simeq \pi$ Howard's [1] asymptotic method based on only two terms in the W expansion yields results in excellent agreement with the former. The following sections of this paper are primarily devoted to the types of solutions that exist when $\gamma \simeq 1$ and $a \gg \pi$. Here the W(z) curve has a "flat top" profile over the greater extent of the fluid layer. These solutions were first noticed by Herring [8] in a numerical study and physically correspond to small elongated convective cells where the vertical velocity reaches a terminal value close to the lower heated boundary. The sine expansion (33), with M large for large R, may still be used to represent the W(z) curve and is a convenient numerical approach. However, our main interest here is to derive analytical solutions for these "flat type" solutions as well as an expression for the total heat transfer involving the appropriate parameters of the problem. The results, involving the use of elliptic functions, are presented in section §2 of this paper, followed in section §3 by a brief description of the numerical method employed and numerical results. Overall, very good agreement is established between the two sets of results.

The linear theory of Chandrasekhar (1961) [9] predicts that for free boundaries, which is the case we consider here, the critical Rayleigh number R_0 is given as a function of the horizontal wave number by (1). Convection will only occur for values of the Rayleigh number

$$R = \frac{g\alpha\Delta T d^3}{\kappa\nu} \tag{2}$$

larger than this value, where g is the acceleration of gravity, α the coefficient of volume expansion, ΔT and d the temperature difference across and depth of the layer, ν the kinematic viscosity and κ the thermal diffusivity.

Quasi-linear or small amplitude solutions for large Rayleigh numbers will only occur in two regions

(i) $a \ll \pi$ if $R = \frac{\pi 6}{a^2 \gamma_1}$ (3)

where $\gamma_1 = 1 - \varepsilon$,

(ii) $a \gg \pi$ if $R = \frac{a_4}{\gamma_2}$ (4)

where $\gamma_2 = 1 - \varepsilon$.

In what follows we shall consider mainly the second case.

2. ASYMPTOTIC METHOD

Solutions of a distinctly non-linear character exist for values of R very near the linear value R_0 when the horizontal wave number a is large. This type of solution is examined in detail in this section. The appropriate equations in the mean-field approximation can be written [3]:

$$(D^2 - a^2)^2 W = Ra^2 F \tag{5}$$

$$(D^2 - a^2)F = WDT_0 , \tag{6}$$

$$DT_0 = FW - N, \text{ where } D \equiv \frac{d}{dz}, \; 0 \leqslant z \leqslant 1 . \tag{7}$$

Eliminating the temperature fluctuation F(z) and the mean temperature $T_0(z)$ from these equations, we obtain the following equation for the vertical velocity W(z),

$$(D^2 - a^2)^3 W = - Ra^2 NW + W^2 (D^2 - a^2)^2 W \tag{8}$$

where N, the Nusselt number, represents the total heat flux across the fluid layer.

Now for large values of a

$$R \simeq a^4 \tag{9}$$

and near linear values of R will again be given by (4). For very large values of R, i.e. of a^4, equation (8) reduces to

$$D^2 W = \beta W^3 - \alpha W \tag{10}$$

where

$$\beta = \frac{1}{3} \tag{11}$$

and

$$\alpha = \frac{a^2}{3} \left(\frac{N}{\gamma_2} - 1\right) . \tag{12}$$

Equation (10) can be solved in terms of Jacobi elliptic functions and a solution can be written as follows

$$W(z) = A \, \mathrm{sn}(\mu z; k) , \tag{13}$$

with

$$\mu = 2K \tag{14}$$

and

$$K = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} . \tag{15}$$

This last expression can also be written

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} \tag{16}$$

and is the complete elliptic integral of the first kind. The amplitude A is given by

$$A^2 = 6(\alpha - \mu^2) \tag{17}$$

and

$$k^2 = \frac{\alpha - \mu^2}{\mu^2} . \tag{18}$$

The properties of the Jacobi elliptic function ensure that the correct boundary conditions

$$W = D^2W = 0 \quad \text{at} \quad z = 0,1 \tag{19}$$

are satisfied.

Combining equations (4) and (5) one has

$$\frac{\gamma_2}{a^6} D^4W - 2\frac{\gamma_2}{a^4} D^2W + \frac{\gamma_2}{a^2} W = F \tag{20}$$

and for very large values of a

$$F = \frac{\gamma_2}{a^2} W . \tag{21}$$

Substituting this value of F from (21) into (7) and integrating from 0 to 1 with respect to z, the following expression for the Nusselt number is obtained

$$N = 1 + \frac{\gamma_2}{a^2} \int_0^1 W^2 dz , \tag{22}$$

or using (4)

$$N = 1 + \sqrt{\frac{\gamma_2}{R}} \int_0^1 W^2 dz . \tag{23}$$

Now the integral

$$\int_0^K sn^2u \, du = \frac{1}{k^2} (K - E) \tag{24}$$

where the complete elliptic integral of the second kind is

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\phi} \, d\phi . \tag{25}$$

Combining (23), (13) and (25) the Nusselt number is given by

$$N = 1 + \frac{A^2}{k^2K} \sqrt{\frac{\gamma_2}{R}} (K - E) . \tag{26}$$

From equation (17) and the definitions of α, μ and β the amplitude can be expressed in the form

$$A^2 = 2a^2\left(\frac{N}{\gamma_2} - 1\right) - 24K^2 . \tag{27}$$

Eliminating A^2 between (26) and (27) means that

$$N = 1 + \frac{1}{k^2K}\left\{2a^2\left(\frac{N}{\gamma_2} - 1\right) - 24K^2\right\} \sqrt{\frac{\gamma_2}{R}} (K - E) . \tag{28}$$

Alternatively from (12)

$$N = \gamma_2\left\{1 + \frac{12}{a^2} (1 + k^2)K^2\right\} \tag{29}$$

and finally eliminating N between these last two equations (28) and (29) gives

$$\gamma_2\left\{1 + \frac{12}{a^2} (1 + k^2)K^2\right\}\left\{1 - \frac{2a^2}{k^2\gamma_2 K} \sqrt{\frac{\gamma_2}{R}} (K - E)\right\}$$

$$= 1 - \frac{2}{k^2K}\left\{a^2 + 12 K^2\right\}\sqrt{\frac{\gamma_2}{R}} (K - E) . \tag{30}$$

This equation is a transcendental equation in k for given values of the parameters R and γ_2. Once this equation has been solved for k the value of the Nusselt number N, velocity amplitude A and temperature fluctuation are given by equations (29), (17) and (21) respectively.

3. NUMERICAL SOLUTIONS

Numerical solutions of the basic non-linear differential equations for this problem have also been obtained over the parameter ranges under discussion in the previous sections i.e. when $\gamma \simeq 1$ and $a >> \pi$. However, in addition we have taken the opportunity to include some general features of these "flat type" solutions which are illustrated by the numerical solutions at parameter values outside the asymptotic values applicable to §2.

Differentiating equation (7) we obtain

$$D^2 T_0 = D(FW) \,, \tag{31}$$

and this equation along with (5) and (6) and the free boundary conditions, given below, define our differential system,

$$\begin{gathered} W = D^2 W = F = 0 \quad \text{at} \quad z = 0 \text{ and } 1 \\ T_0 = 0 \text{ at } z = 0 \quad \text{and} \quad T_0 = -1 \text{ at } z = 1 \,. \end{gathered} \tag{32}$$

The following truncated Fourier representations of the variables were used

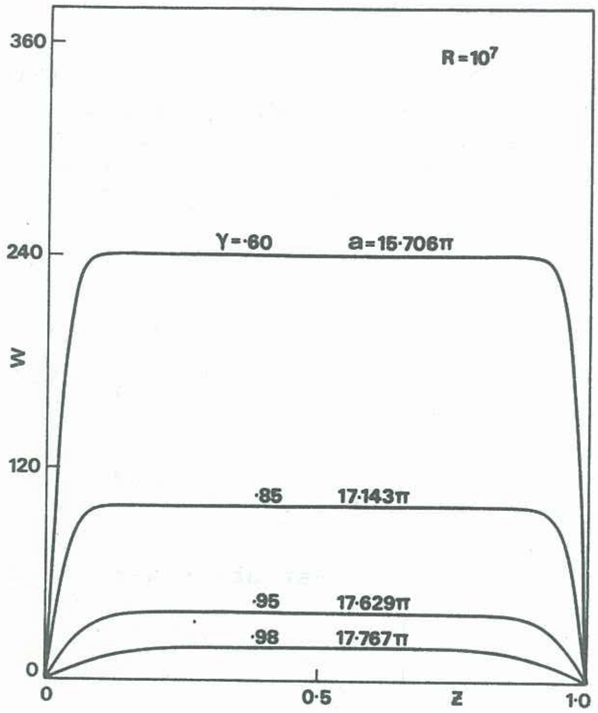
$$\left.\begin{aligned} W(z) &= \sum_{n=1}^{M} W_n \sin[(2n-1)\pi z] \\ F(z) &= \sum_{n=1}^{M} f_n \sin[(2n-1)\pi z] \\ T_0 &= \sum_{n=1}^{M} t_n \sin(2\pi n z) - z \end{aligned}\right\} \tag{33}$$

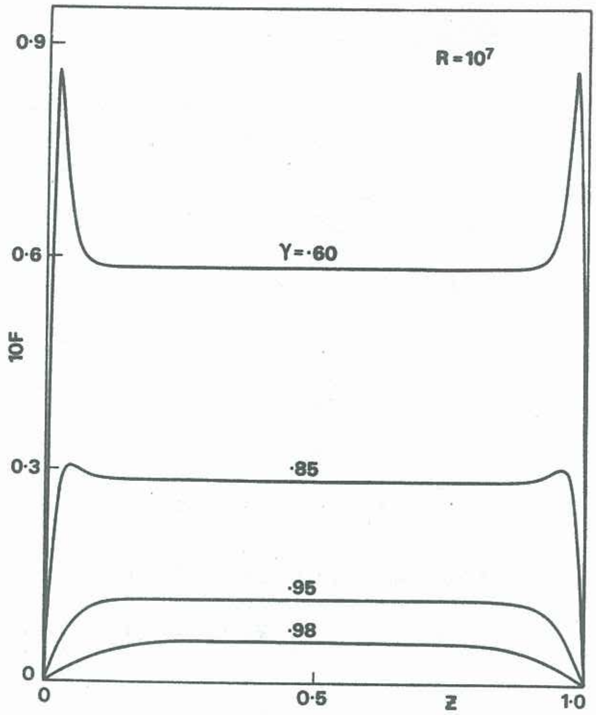
and on substitution into (5), (6) and (31) lead to a system of non-linear equations for the coefficients W_n, f_n and t_n $(n = 1,2,\ldots M)$. These equations were solved by the application of the generalized Newton-Raphson method. The value of M, the number of terms included in the expansions (33), was adjusted to ensure convergence of the solutions and physically represents the criterion of constant heat flux.

Figure 1 shows the variation across the layer of $W(z)$, $F(z)$ and $T_0(z)$, $0 \leqslant z \leqslant 1$, when $R = 10^7$ for varying γ. In each case the vertical velocity is seen to be constant, having reached a terminal value, across the greater part of the layer. This constancy of W is associated with a linear dependence on T_0 and constant F for the values of γ considered. However, it can be noted that there are sharp changes in F near the free boundaries when $\gamma = .60$ - a characteristic of the $a \simeq \pi$ solutions. As γ varies from .60 to .98 the non-linearity of the solutions is well illustrated when one compares the W solution with the linear sine solution. In fact when $a = 17.768\pi$ $(\gamma = .98)$ the ratio $W_2/W_1 = .456/2.284$, and this must be compared to the linear solution at $a = 17.858\pi$ $(\gamma = 1.00)$.

4. CONCLUSIONS

Non-linear small amplitude solutions for large R have been shown to exist for $a >> \pi$ when γ is very near the linear value $\gamma = 1.00$, and the exact nature of these solutions has been illustrated by the numerical results from §3. It is now of interest to compare the asymptotic and numerical approaches. In figure 2 the results for the Nusselt number, defining the total heat flux, are given as a function of γ for several values of R. The continuous line gives the N dependence determined from section §2 and the corresponding values of N from the direct numerical method are shown as individual points; clearly there is very good agreement in the values. The triangles on this diagram are for values of $k^2 \simeq 1$ and indicate the extent of the asymptotic method. The corresponding value of γ for $R = 10^7$ is 0.94, hence only the $\gamma = .95$ and .98 curves from figure 1 can be taken as representative of the asymptotic solutions derived in section §2. In table 1 a further comparison between the two sets of values has been made for N, W at $z = \frac{1}{2}$ and F at $z = \frac{1}{2}$ - again good agreement is evident.





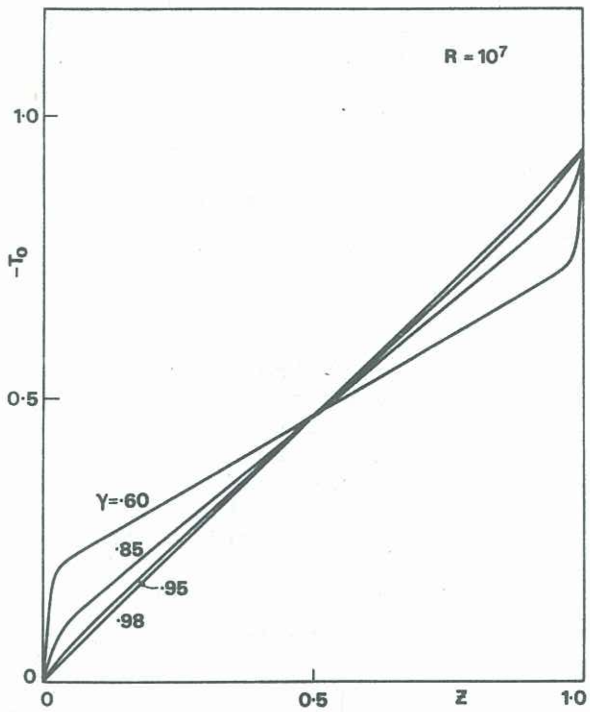


Figure 1: Numerical results for W, F and T_0 across the fluid layer, $0 \leqslant z \leqslant 1$, when $R = 10^7$ for the indicated values of γ.

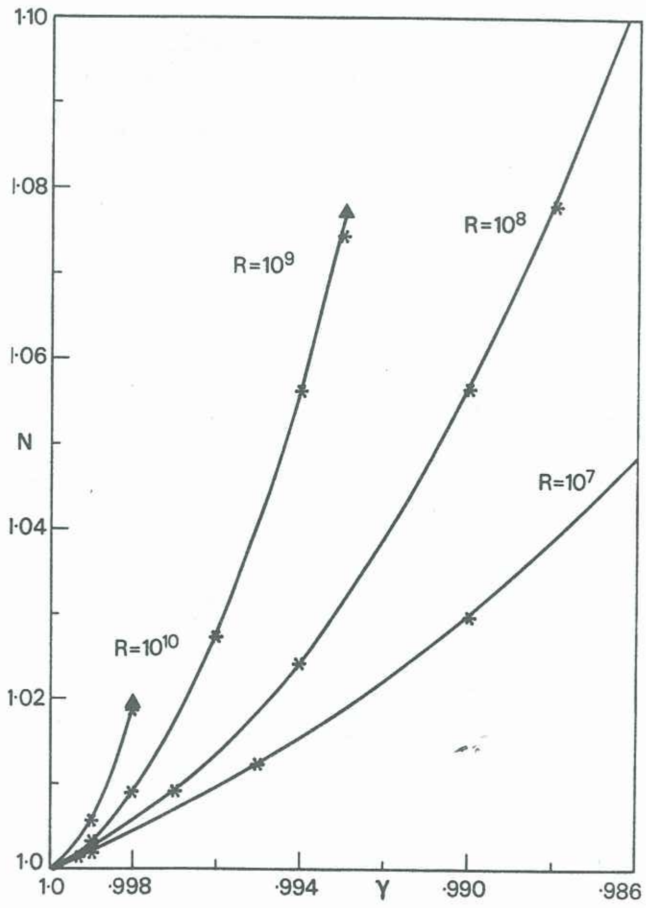


Figure 2: The dependence of the Nusselt number N on γ is illustrated for four values of the Rayleigh number R. The results from section §2 are represented by the continuous line and the ▲ correspond to the values of γ for $k^2 = 1$. The values of N computed from section §3 are shown as individual points (*).

TABLE 1

A comparison of the values obtained from the asymptotic formulation (subscript E) and the numerical solutions, when $R = 10^7$, for the indicated values of γ, for the Nusselt number N, temperature fluctuation F at $z = \frac{1}{2}$ and the maximum value of the vertical convective velocity W.

$R = 10^7$ $k^2 \simeq 1$ when $\gamma = .940$

γ	k^2	$a^2/10^3$	N_E	N	$10^2.F(\frac{1}{2})_E$	$10^2.F(\frac{1}{2})$	$W(\frac{1}{2})_E$	$W(\frac{1}{2})$
.999	.2006	3.1459	1.0021		.114		3.643	
.997	.4916	3.1427	1.0068		.199		6.345	
.995	.6833	3.1395	1.0123	1.0122	.260	.260	8.308	8.258
.990	.9095	3.1316	1.0301	1.0298	.384	.383	12.267	12.202
.985	.9756	3.1237	1.0543		.492		15.765	
.980	.9936	3.1157	1.0851	1.0833	.596	.591	19.135	18.975
.975	.9983	3.1077	1.1228		.698		22.474	
.970	.9995	3.0997	1.1674	1.1692	.800	.805	25.816	25.974
.965	.9998	3.0916	1.2189		.901		29.171	
.960	.9999	3.0836	1.2773	1.2733	1.003	.998	32.541	32.390
.955	.9999	3.0755	1.3429		1.105		35.930	
.950	.9999	3.0674	1.4155	1.3888	1.206	1.172	39.337	38.218
.945	.9999	3.0592	1.4952		1.308		42.760	
.940	.9999	3.0511	1.5822		1.409		46.204	

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