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AN EXTENSION TO THE THEORY OF SELECTIVE WITHDRAWAL

by

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SUMMARY

The particular case of selective withdrawal from a two layered reservoir via a horizontal and vertical contraction to a valved conduit with a withdrawal point above the level of the bottom of the reservoir is considered. In the theoretical development it is assumed that the layers in the reservoir are stable and that the flow is inviscid and gradually varied. The case when both layers are flowing and the discharge is determined by the valve is considered. For this case a solution which gives the proportion of discharge coming from each layer is obtained using a simple extension of open channel flow theory. Experiments are described which verify the theory and show that when the gradually varied assumption breaks down the theory is no longer effective. An empirical transformation is suggested in order that the theory may be used for these cases.

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INTRODUCTION

Vertical density gradients exist in most of man's water resources (rivers, lakes and oceans) because of the variations in depth of temperature salinity and turbidity. In managing these resources it is often advantageous to selectively withdraw water of an appropriate quality. The most common example of this is the case of the withdrawal of the coolest water for the condenser system of thermal power stations. Indeed considerable fuel savings can be made by using the coolest possible water for this system. For some conditions when the stratified fluid consists of essentially two layers it is possible to withdraw only the coolest possible fluid. However, when the discharge exceeds a critical value then some of the warmer overlying fluid is also carried into the cooling system and in this case engineers look to predict the quality (i.e. temperature) of the cooling fluid.

For the case where the withdrawal slot was at the base of reservoir and where the flow into the slot could be assumed inviscid and gradually varied a theory was developed by Wood and Lai (1). The work in this paper extends this theory and suggests an empirical transformation which appears to enable the theory to be used for the more practical cases where the flow is strongly curvilinear.

Theory. Consider a reservoir with a horizontal and vertical contraction as in Figure (1). Let it be assumed that the contraction is symmetrical about the ox axis and let the width of the contraction be defined by

$$\frac{b}{b_0} = e^{K\left(\frac{x}{b_0}\right)^2} \qquad (1)$$

where x is the distance from the minimum width of the contraction, b_0 and b are the total breadths of the contraction at the origin and x respectively and K is a constant. In this paper only small values of K will be dealt with and this implies that variations in the width of the contraction are gradual. Let the contraction be symmetrical about the ox axis and let the height of the contraction be defined by

$$p - p_0 = \beta \log_e \left(\frac{b}{b_0}\right) \qquad (2)$$

where p_0 and p are the total heights of the contraction at the origin and x respectively and β is a constant. Small values of β imply that the variation of heights of the contraction are gradual while large values imply a rapid change with x. This definition of p and b gives a convenient value for $\frac{dp}{db}$ and this will be used later in the theory. Consider the case of a two layer flow through the contraction as shown in Figure (1).

In this particular case the interface is above the X axis. Let the depth of layer 1 in the reservoir be Y_1 and that of layer 2 be Y_{2u} above the axis and Y_{2L} below the axis. Further let the density of the atmosphere be ρ_0 and of the fluid in layer 1 be $\rho_1 = \rho_0+\Delta\rho_1$ and in layer 2 $\rho_2=\rho_0+\Delta\rho_1+\Delta\rho_2$. Let y_{2u} and y_{2uL} be the depths of the layer 2 above and below the axis at any x. Similarly let y_1 be the depth of layer 1. Let the pressure on the upper surface on the contraction be h.

If the flow is gradually varied such that the pressures in the fluid are to be a good approximation of hydrostatic and if the horizontal curvature is sufficiently gradual such that the one dimensional approximation can be used then simple extensions of the open channel flow theory can be used to obtain the conditions for incipient drawdown and the quality of the water when a proportion of the upper layer is being withdrawn. The conditions for incipient drawdown of the upper layer can be obtained following the method of Wood and Lai (1) and need not concern us here. In this paper attention will be restricted to the case when both layers are flowing. For this case Bernoulli's equation in the lower layer is

$$\frac{1}{2}\rho_1 \left(\frac{Q_1}{by_1}\right)^2 + \Delta\rho_1 g\,(h+y_1+y_{2u}+y_{2L}+H) = \Delta\rho_1 g\,(Y_1+Y_{2u}+Y_{2L}) \qquad (3)$$

where H is as in Figure (1). Since $y_{2L} + H = Y_{2L}$ and if we define $H_1 = y_1 + h$, then this equation becomes

$$\frac{1}{2}\frac{\rho_1}{\Delta\rho_1 g}\left(\frac{Q_1}{by_1}\right)^2 + H_1 + y_{2u} = Y_1 + Y_{2u} \qquad (4)$$

Bernoulli's equation for layer 2 may be written

$$\frac{1}{2}\frac{\rho_2}{\Delta\rho_2 g}\left(\frac{Q_2}{by_2}\right)^2 + \alpha H_1 + (\alpha+1)\, y_{2u} = \alpha Y_1 + (\alpha+1)\, Y_{2u} \tag{5}$$

where $\alpha = \frac{\Delta\rho_1}{\Delta\rho_2}$ and H_1 is as before.

Differentiating the above equations and defining the Froude Nos. as

$$F_1^2 = \frac{\rho_1}{\Delta\rho_1 g}\left(\frac{Q_1}{by_1}\right)^2$$

$$F_2^2 = \frac{\rho_2}{\Delta\rho_2 g}\left(\frac{Q_2}{by_2}\right)^2$$

We get

$$-F_1^2\,\frac{y_1}{b}\frac{db}{dx} - F_1^2\,\frac{dy_1}{dx} + \frac{dH_1}{dx} + \frac{dy_{2u}}{dx} = 0 \tag{6}$$

and

$$-F_2^2\,\frac{y_2}{b}\frac{db}{dx} - F_2^2\,\frac{dy_2}{dx} + \frac{\alpha dH_1}{dx} + (\alpha+1)\,\frac{dy_{2u}}{dx} = 0 \tag{7}$$

Now $y_2 = y_{2L} + y_{2u}$ which gives

$$\frac{dy_{2u}}{dx} = \frac{dy_2}{dx} - \frac{dy_{2L}}{dx} \tag{8}$$

and $p = y_1 + y_2$ which gives

$$\frac{dy_1}{dx} = \frac{dp}{dx} - \frac{dy_2}{dx} \tag{9}$$

Using equations (8) and (9), and substituting for $\frac{dp}{db}$ and $\frac{dy_{2L}}{db}$ as given by manipulating equations (1) and (2), equations (6) and (7) become respectively

$$\frac{dH_1}{dx} + (1+F_1^2)\,\frac{dy_2}{dx} = \frac{1}{b}\frac{db}{dx}\left(F_1^2\,(y_1+\beta) + \frac{\beta}{2}\right) \tag{10}$$

$$\alpha\,\frac{dH_1}{dx} + (1+\alpha-F_2^2)\,\frac{dy_2}{dx} = \frac{1}{b}\frac{db}{dx}\left(F_2^2\,y_2 + (1+\alpha)\,\frac{\beta}{2}\right) \tag{11}$$

The simultaneous equations (10) and (11) can be solved for $\frac{dy_2}{dx}$, giving

$$\frac{dy_2}{dx} = \frac{1}{b}\frac{db}{dx}\frac{D_2}{D_0} \tag{12}$$

where

$$D_0 = 1 - F_2^2 - \alpha F_1^2 \tag{13}$$

and

$$D_2 = F_2^2\,y_2 - \alpha F_1^2\,(y_1+\beta) + \frac{\beta}{2} \tag{14}$$

At this stage it is convenient to assume that the depth in layer 1 is large. This ensures that F_2^2 and αF_1^2 are small at the commencement of the vertical contraction. This ensures that D_0 is greater than zero at this point and is in fact close to unity. At the minimum width of the contraction fluid velocities will be high if F_1^2 and F_2^2 are large and thus D_0 will be less than zero. There is thus a point in the closed conduit region where $D_0 = 0$. This point is called the virtual control (and is labelled V in Figure (2). To meet the requirements that $\frac{dy_2}{dx}$ is finite and positive it is required that $D_2 = 0$ when $D_0 = 0$ and D_2 has the same sign as D_0 in the closed conduit region. Thus D_2 must change sign across the virtual control. The method of solution is via the equations derived at the virtual control. (It can be shown that for the conditions discussed above the conditions for a control in the region upstream of the vertical contraction ($D_0' = (1-F_1^2)(1-\alpha-F_2^2)-\alpha = 0$ (2)) cannot be satisfied). Equations (4) and (5) may be written as

$$\frac{1}{2}F_1^2\,y_1 + T = Y_1 + Y_{2u} + y_{2L} \tag{15}$$

$$\frac{1}{2}F_2^{\,2}\,y_2 + \alpha T + y_2 \quad = \quad \alpha Y_1 + (1+\alpha).(Y_{2u}+y_{2L}) \tag{16}$$

where $T = y_1 + y_2 + h$. Eliminating T from these equations gives

$$\frac{1}{2}\left(\alpha F_1^{\,2}\,y_1 - F_2^{\,2}\,y_2\right) - y_2 \quad = \quad -Y_{2u} - y_{2L} \tag{17}$$

At the virtual control therefore there are three equations (13), (14) and (17). Noting that $y_{2L} = p/2$ and $y_1 = p-y_2$ and defining

$$\varphi = \frac{1}{2}\,\frac{\rho_1}{\Delta\rho_1 g}\,\left(\frac{Q_1}{by_1}\right)^2 \tag{18}$$

$$c^2 \quad = \quad \frac{\rho_2 v_2^{\,2}}{\rho_1 v_1^{\,2}} \quad = \quad \frac{\rho_2}{\rho_1}\left(\frac{by_1}{Q_1}\right)^2\left(\frac{Q_2}{by_2}\right)^2 \tag{19}$$

then equation (17) becomes

$$\alpha\varphi\,(1-c^2) - y_2 \quad = \quad -\,Y_{2u} - \frac{p}{2} \tag{20}$$

and $D_0 = 0$ becomes

$$1 - \frac{2\alpha\varphi c^2}{y_2} - \frac{2\alpha\varphi}{p-y_2} \quad = \quad 0 \tag{21}$$

and $D_2 = 0$ becomes

$$2\alpha\varphi c^2 - \frac{2\alpha\varphi}{p-y_2}\,(p-y_2+\beta) + \frac{\beta}{2} \quad = \quad 0 \tag{22}$$

We thus have three equations and four unknowns. The fourth equation comes from the discharge obtained from the valve. Thus one of the unknowns may be selected.

If Q_{21} is defined as

$$Q_{21} \quad = \quad \frac{\sqrt{\rho_2}\,Q_2}{\sqrt{\rho_1}\,Q_1} \quad = \quad \frac{cy_2}{y_1} \tag{23}$$

then after some algebra φ and y_2 and p can be eliminated from these equations giving us

$$Q_{21}^{\,3}\;{}^{2}\beta + Q_{21}^{\,2}\Big[[27\beta+8Y_{2u}]\,c^3 + c[-4\beta+8Y_{2u}]\Big] + Q_{21}\,[c^6\,[-4\beta+8Y_{2u}] + c^4\,[2\beta+8Y_{2u}]] + 2\beta c^7 \quad = \quad 0 \tag{24}$$

If a value of c is selected than a solution for Q_{21} can be obtained from this cubic. The appropriate solution is the largest root of the cubic. An equation of exactly similar form can be developed for the case where the interface is below the axis. A typical result is shown in Figure (4) where for a fixed discharge the ratio (Q_{21}) is plotted against the level of the lower layer in the reservoir.

EXPERIMENTS

The experimental apparatus is shown in Figure (3). The horizontal contraction was given by the equation,

$$\frac{b}{b_0} \quad = \quad e^{.0052\,\left(\frac{x}{b_0}\right)^2} \tag{25}$$

where $b_0/2 = 1.00''$. The relationship between the vertical and horizontal contraction was given by

$$p - p_0 = \beta\log_e\left(\frac{b}{b_0}\right) \tag{26}$$

where $p_0 = 2.54$cm. For a contraction in which the gradually varied flow assumption was satisfied the value of β was seven and for the case where the flow was markedly curvilinear β was infinite. In each case the withdrawal slot contained a 15.24cm straight section. The slot was connected to a clear plastic hose which led to a tap which in turn ran off into a drain.

After the set up of two levels in the reservoir the experiment was started by opening the tap controlling inflow of fresh water and the valve controlling the outflow of the mixture. At intervals readings were taken of the height of the interface in relation to the centreline of the contraction and discharge measurements were taken. In addition a sample of the discharge was taken and readings were continued until the interface dropped down such that virtually no salt solution was being discharged. At this stage the experiment was over. Photographs of two typical experiments are shown in Figure (6). By maintaining the upper level in the reservoir at a constant level the discharge was almost constant throughout the experiment. By analysing the salinity of the collected samples the ratio of the discharge from the upper and lower layer could be determined. A typical comparison of experiment and theory for the case of a gradually varied contraction is shown in Figure (4). A similar comparison of theory and experiment is shown for the vertical wall in Figure (5). It is clear that in this case and indeed for all cases of steep contraction that the theory overestimates the range of values of Y_2/p_0 for which selective withdrawal of both layers would occur. This suggests that in situations of steep contractions the flowing layers establish their own control. A simple empirical transformation which enables the theoretical results to be transformed to give results which fit the experimental values is

$$\left(\frac{Y_2}{p_0}\right)^T = 0.5\left(\frac{Y_2}{p_0} + \left(\frac{Y_2}{p_0}\right)_{cL}\right) \tag{27}$$

where $\left(\frac{Y_2}{p_0}\right)_{cL}$ is the value of Y_2/p_0 at the centre line. The effect of this transformation is shown in Figure (5). It must be emphasised that although similar agreement was obtained for all experiments with $\beta = \infty$ and for other steep contractions ($\beta = 56$) the transformation is an empirical one. At the present stage no physical explanation for the success of this transformation can be given.

REFERENCES

(1) Wood, I.R. and Lai, K.K., "Selective Withdrawal from a Two-Layered Fluid", Journal of Hydraulic Research, Vol. 10, No. 4, pps. 475-496, 1972.

(2) Woood, I.R., "Selective Withdrawal from a Stably Stratified Fluid", J. Fluid Mech. Vol. 32, Pt. 2, pp. 209-223, 1968.

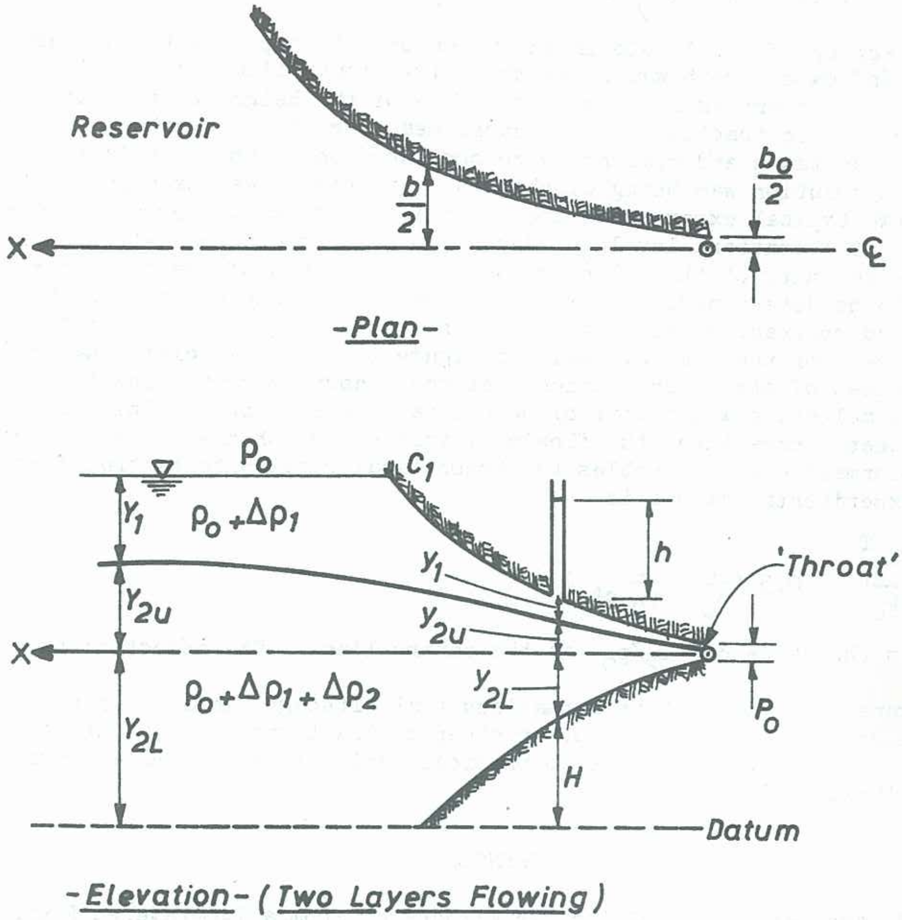

Fig.1: NOMENCLATURE INTERFACE ABOVE AXIS

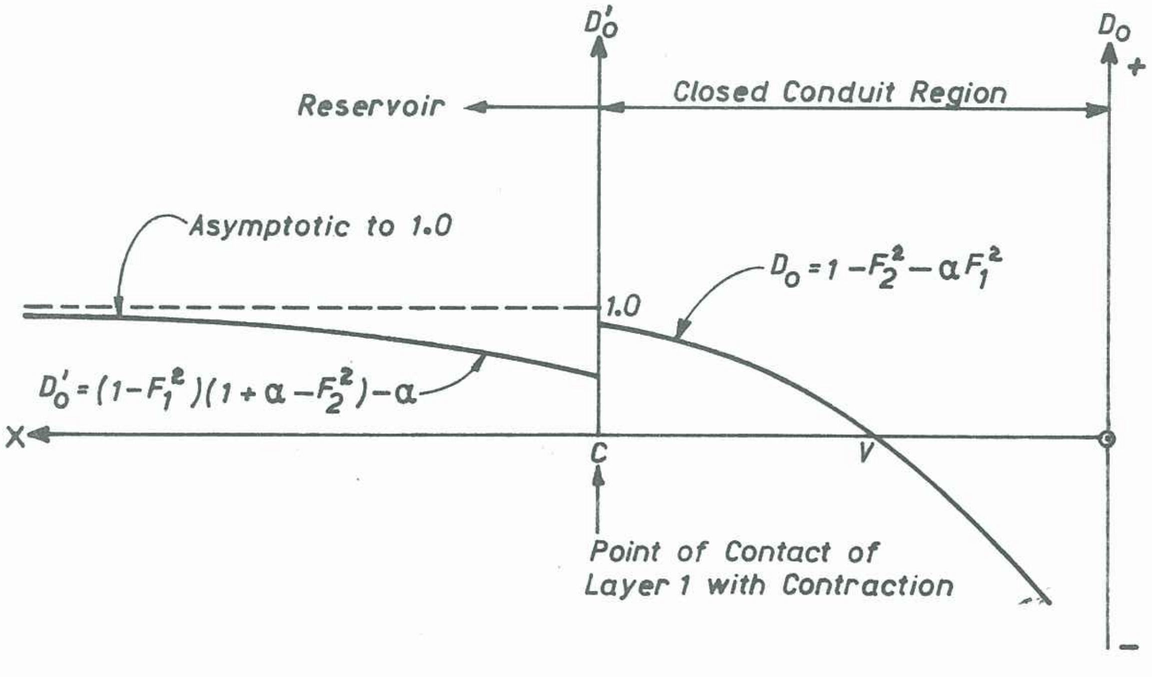

Fig. 2: GRAPH OF D_o AND D_o' VERSUS X

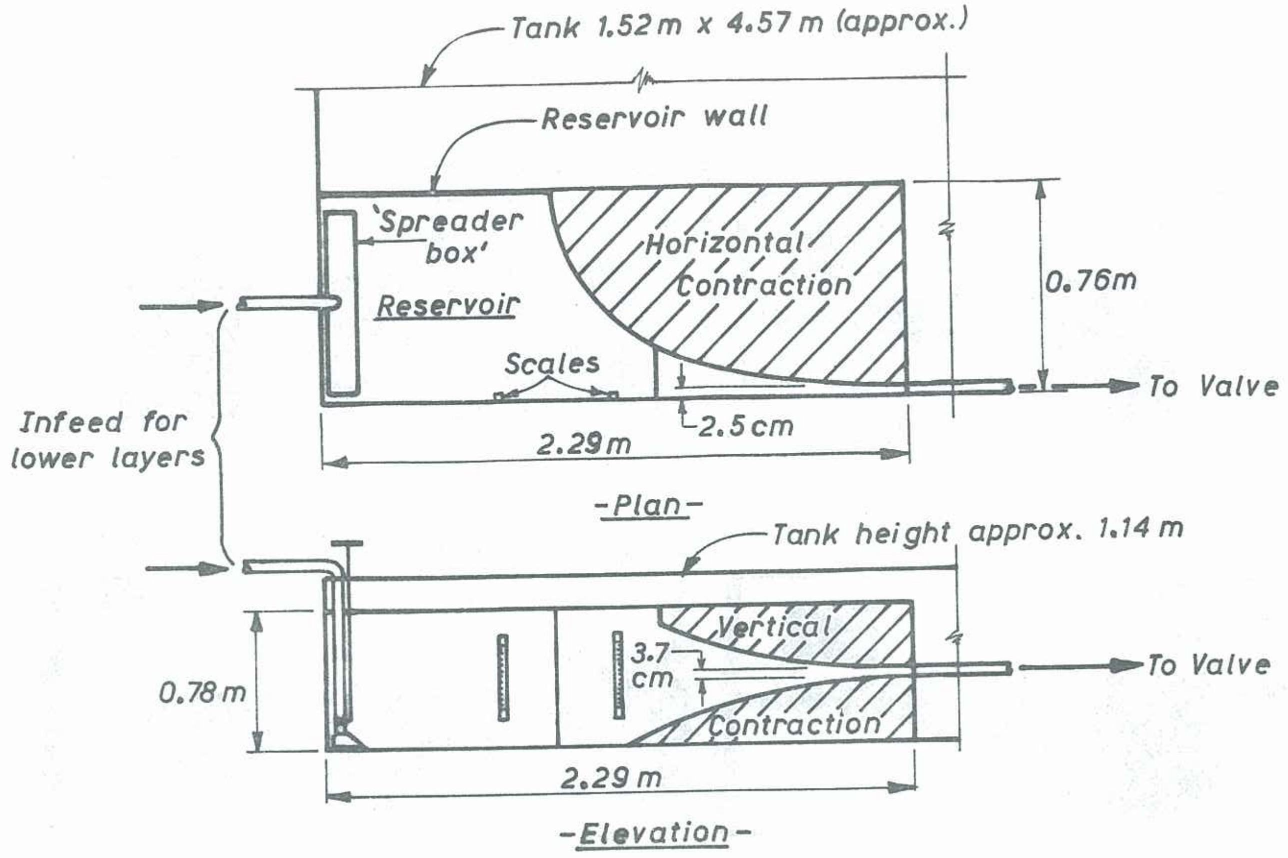

Fig. 3: THE EXPERIMENTAL EQUIPMENT

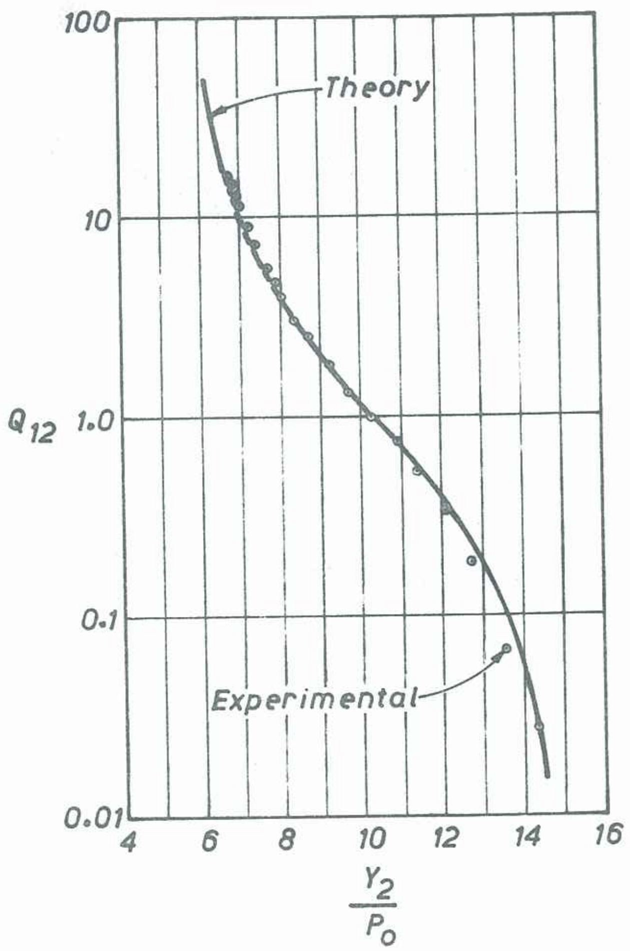

Fig. 4 : (α = 400, β = 7.0)

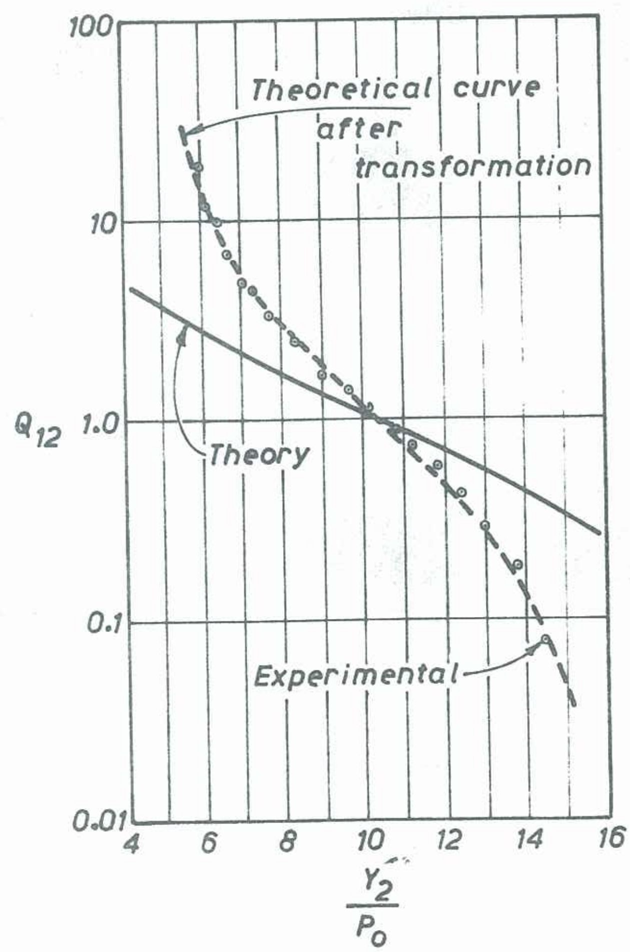

Fig.5 : (α = 385, β = Infinite)

- COMPARISON OF THEORETICAL & EXPERIMENTAL RESULTS -

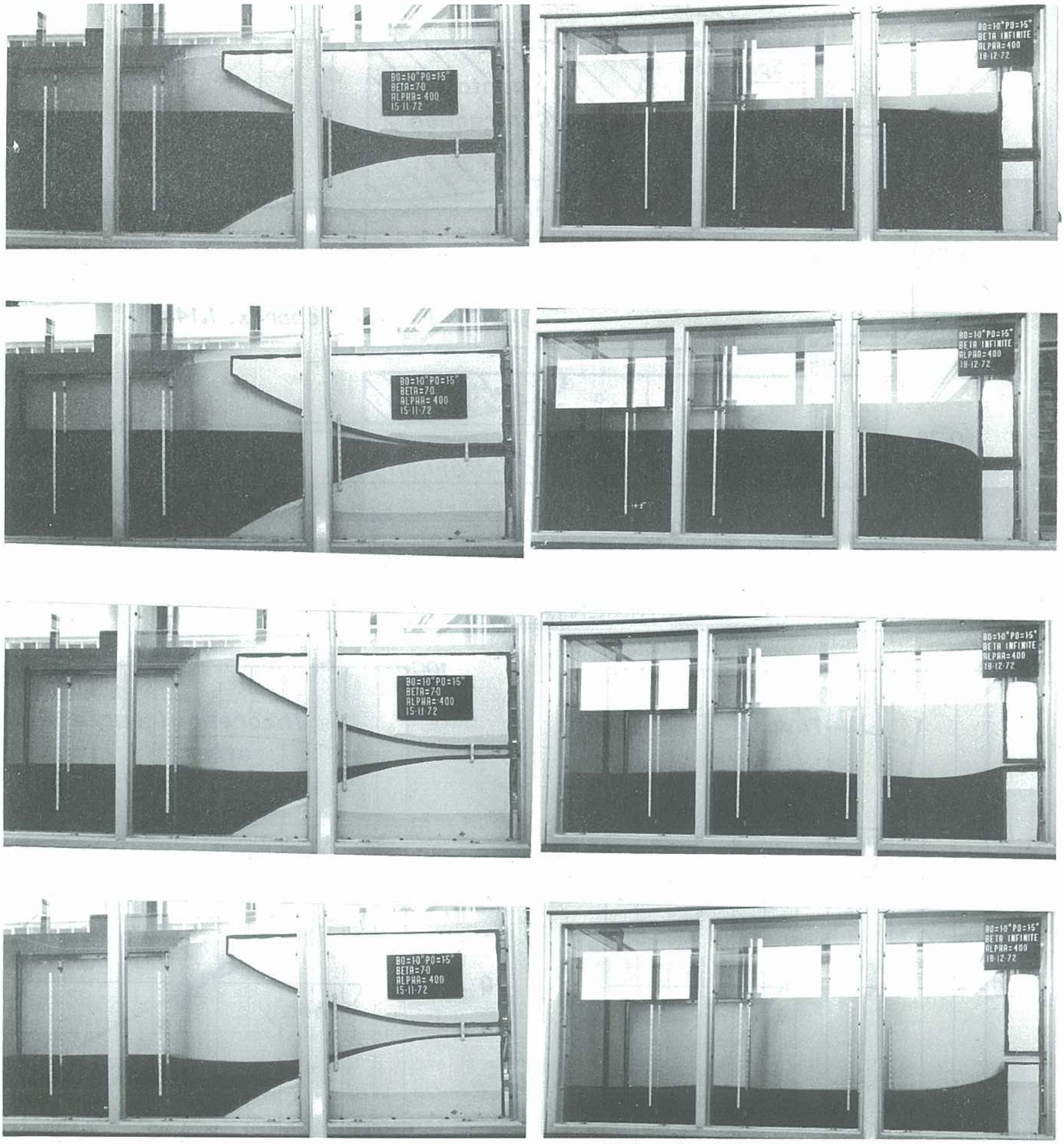

Fig. 6 : RECORDS OF TYPICAL EXPERIMENTS

(a) Contraction with gradually varied flow (b) Vertical contraction