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Mixing Of Two Streams At Different Densities

by

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SUMMARY

The flow of a stream of viscous incompressible fluid over a parallel stream of different density and viscosity was examined theoretically. It was shown that the density profiles across the interfacial boundary layer were much thinner for salt transfer than for heat transfer. The theoritical density profiles were in excellent agreement with the measurements by Lofquist. Complications of applying the theoretical results to Duwamish River data were also discussed.

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INTRODUCTION

Salt wedges are likely to form in river estuaries under certain flow conditions and eventually to extend several miles upstream, where the riverbed is fairly even and the tide is not strong enough for sudden flow reversals to take place. They are often observed in the lower reaches of the Mississippi River, particularly in the seasons of reduced river discharges.

The salt wedge consists of one layer of freshwater flowing over a layer of saline water, the two separated by an interface. The interface is a layer of transition between two currents. Both the densities and the velocities change in a layer that has measurable thickness. Two modes are frequently observed: either the interface remains distinct or mixing takes place between the two layers and destroys the interface completely. The limiting conditions responsible for the formulation of a distinct interface occur when the tidal velocities and mean flow velocities are unable to provide sufficient energy to the turbulence to overcome the effects of the density stratification to become unstable and break, so that the exchanges between the river water and the sea water are essentially molecular and occur through a laminar boundary layer. Thus the flow can be considered separately in the two layers with a matching to be made through the laminar boundary layer.

The problem of determining the velocity distribution between two parallel streams has been considered by Lock (1) and satisfactory results have been obtained. It is the purpose of this investigation to determine the density distribution when the parallel streams are of different densities.

FORMULATION OF THE PROBLEM

Our model simulating the interface formed in the river estuary consists of freshwater at velocity U_1 flowing over saline water at velocity U_2 (Fig. 1). The x-axis is in the direction of river flow, the y-axis points upward, with the origin located at the interface. The cross-channel velocity is considered small compared with the x-component and is neglected here. In addition, the flow is assumed steady and free of boundaries. These assumptions are considered reasonable as long as the formation of the interface under consideration is sufficiently removed from the boundaries. The "steady" assumption is valid if the flow variation is small during the time interval of interface formation. Even with the preceding assumptions, we are still faced with the nonlinear partial differential equations in u, v, T with respect to the independent variables x and y. Fortunately, the equations can be simplified by invoking the boundary layer approximation. This approximation is based on the assumption that the large velocity gradients $\partial u/\partial y$ occurs only within a narrow region. In the case of our mixing layer, it is the laminar mixing region in which the very small viscosity μ of the fluid exerts an essential influence whenever the shearing stress $\tau = \mu(\partial u/\partial y)$ assumes large values. In the remaining region no such large velocity gradients occur, and the influence of viscosity is unimportant. The same argument is applied to the mass-transport equation, whereas the viscosity is replaced by the mass-diffusion coefficient. Based on the boundary layer argument (for detail see Schlichting (3), it can be shown that the equations are reduced to the following

$$u u_x + v u_y = \nu u_{yy} \tag{1}$$

$$u T_x + v T_y = k T_{yy} \tag{2}$$

$$u_x + v_y = 0 \tag{3}$$

for the upper and lower fluid(Fig.1). Here T may be considered as temperature or salinity, depending on the value of diffusivity, and the fluid density will be treated as a function of T only, where ν is the kinematic viscosity. Because the momentum equation is decoupled from the temperature T, the velocity field may be determined independently of the temperature field.

Since the system under consideration has no preferred length, it is reasonable to suppose that the velocity profiles within the mixing layer at varying distances from the point of contact of the two layers are similar to each other, which means that the velocity u(y) for varying distance x can be made identical by selecting scale factors for u and y. The scale factors u and y appear naturally as the free stream velocity U_1 and the boundary layer thickness $\delta_1(x)$, respectively, for the upper fluid. Hence the principle of similarity of velocity profiles in the mixing layer can be written as $u/U_1 = f_1'[y/\delta_1(x)]$ where $\delta_1(x) = (U_1/\nu_1 x)^{-1/2}$. The continuity equation (3) already has been satisfied by introducing the stream function f_1 for the upper fluid and f_2 for the lower fluid. It may be remarked that the "similarity hypothesis" is based on the lack of preferred length. Hence, in reality it ceases to be valid when the downstream distance x is of the same order as the thickness of either layer.

Accordingly, the momentum boundary layer equations are reduced to

$$u = U_1 f_1'(\eta_1), \quad \eta_1 = (U_1/\nu_1 x)^{1/2} y,$$

$$2\frac{d^3 f_1}{d\eta_1^3} + f_1 \frac{d^2 f_1}{d\eta_1^2} = 0 \tag{4}$$

for the upper fluid and

$$u = U_1 f_2'(\eta_2), \quad \eta_2 = (\nu_1/\nu_2)^{1/2} \eta_1$$

$$2\frac{d^3 f_2}{d\eta_2^3} + f_2 \frac{d^2 f_2}{d\eta_2^2} = 0 \tag{5}$$

for the lower fluid.

And the boundary conditions are

$$u \to U_1, \quad \eta_1 \to \infty \;; \qquad u \to U_2, \quad \eta_2 \to -\infty,$$

or

$$f_1' \to 1, \quad \eta_1 \to \infty, \qquad f_2' \to U_2/U_1, \quad \eta_2 \to -\infty. \tag{6}$$

The continuity of velocity and tangential stress at the interface are

$$f_1'(0) = f_2'(0) \tag{7}$$

and

$$\rho_1 \nu_1^{1/2} f_1''(0) = \rho_2 \nu_2^{1/2} f_2''(0) \tag{8}$$

Integration of the differential equations (4) and (5) together with matching conditions (7) and (8) are quite involved, only the results are given below:

$$\frac{T_1 - T}{T_1 - T_2} = \frac{\int_0^1 (\cos\frac{\pi}{2}t)^\sigma dt - \int_0^{\eta_1} (\cos\frac{\pi}{2}t)^\sigma dt}{\int_0^1 (\cos\frac{\pi}{2}t)^\sigma dt - \int_0^{-1} (\cos\frac{\pi}{2}t)^\sigma dt}, \quad 0 \leq \eta_1 \leq 1 \qquad (9)$$

$$\frac{T_1 - T}{T_1 - T_2} = \frac{\int_0^1 (\cos\frac{\pi}{2}t)^\sigma dt - \int_0^{\eta_2} (\cos\frac{\pi}{2}t)^\sigma dt}{\int_0^1 (\cos\frac{\pi}{2}t)^\sigma dt - \int_0^{-1} (\cos\frac{\pi}{2}t)^\sigma dt}, \quad -1 \leq \eta_2 \leq 0 \qquad (10)$$

The formulae (9) and (10) are approximate when the density ratio of the two streams is close to unity. This is the case for an estuary.

DISCUSSION OF THE RESULTS

The density profiles across the boundary layer are presented in Fig. 2 for σ = 1, 7, 100, 500, 600, 700. If the density difference derives from the temperature difference, then σ =1 is the case for air σ = 7 for water. It the density difference derives from the salinity difference, then a typical value of σ_1 is 684, based on the values of

$$\nu_1 = 9.58 \times 10^{-3}\ cm^2/sec \quad (23°C,\ 0.306\ m\ell^{-1})$$

$$\kappa_1 = 1.40 \times 10^{-5}\ cm^2/sec \quad (23°C,\ 0.306\ m\ell^{-1})$$

Thus it is to be expected that the density thickness is much thinner for salt transfer than for heat transfer.

Very few experiments have been carried out in the case of laminar mixing between fresh water and saline water. The only experiments of this nature were done by Lofquist(2). In Fig. 3 the density profile is plotted against the depth normalized by the density thickness $\delta = [\Delta\rho / -(\partial\rho/\partial z)_m]$. In the same figure, the empirical density profile

$$\frac{\rho - \rho_f}{\Delta\rho} = \frac{1}{2}\left[1 - \tanh\frac{z-h}{\delta/2}\right]$$

proposed by Lofquist,also has been given. Their agreement is good through most of the depth except for values of $(z-h)/\delta/2 > 1.0$. This is reasonable because Lofquist's curve simulates the combined profiles including a large number of turbulent cases. Hence the laminar density profile given here should be consistently thinner than that of the turbulent case.

One interesting result of this study is the ratio of the density thickness to the momentum boundary thickness, which is about 1/20 for σ =700. Unfortunately, Lofquist's measurements of the velocity and density were made at two different locations; hence a comparison is difficult. However, according to Lofquist(Private communication), the measurements of density and velocity profiles were made at stations only 60 cm. apart, or only 2.5% of the length of the flow. Based on the density and velocity profiles given by Lofquist(2), the ratio of the density thickness to the velocity layer thickness is approximately 1/10. Two comments are in order. First, it was difficult to estimate the boundary layer thickness for both density and velocity. Second, the Lofquist's data represent cases of both turbulent and laminar .

Application of the preceding results to natural estuary system is difficult. In recent years, U.S.Geological Survey collected both velocity and salinity data in the Duwamish River, Seattle, Washington. Some of the data are presented in Figure 4 and Figure 5. It is clear that the steady state assumption is in doubt. It is also well known that mixing coefficient is dependent on the degree of density stratification and velocity shear. Therefore the estuary flow problem is a coupled one. Current efforts were devoted to compute eddy mixing coefficients from the measurements.

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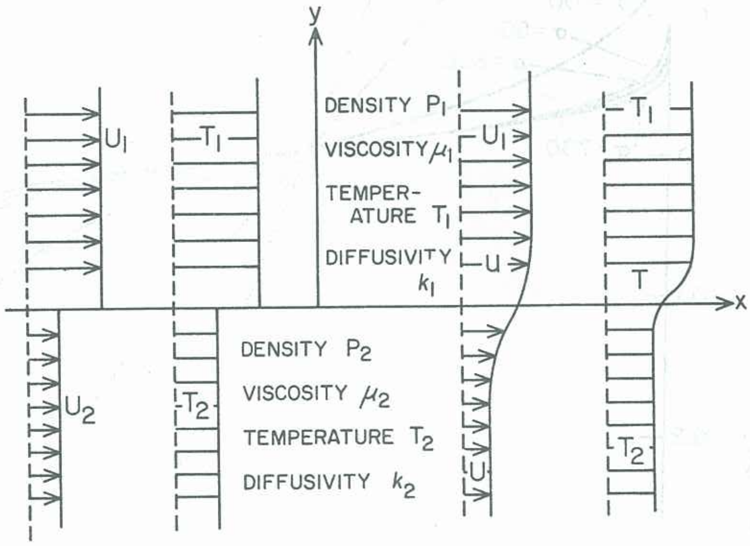

FIG. 1.—MIXING OF TWO SEMI-INFINITE STREAMS.

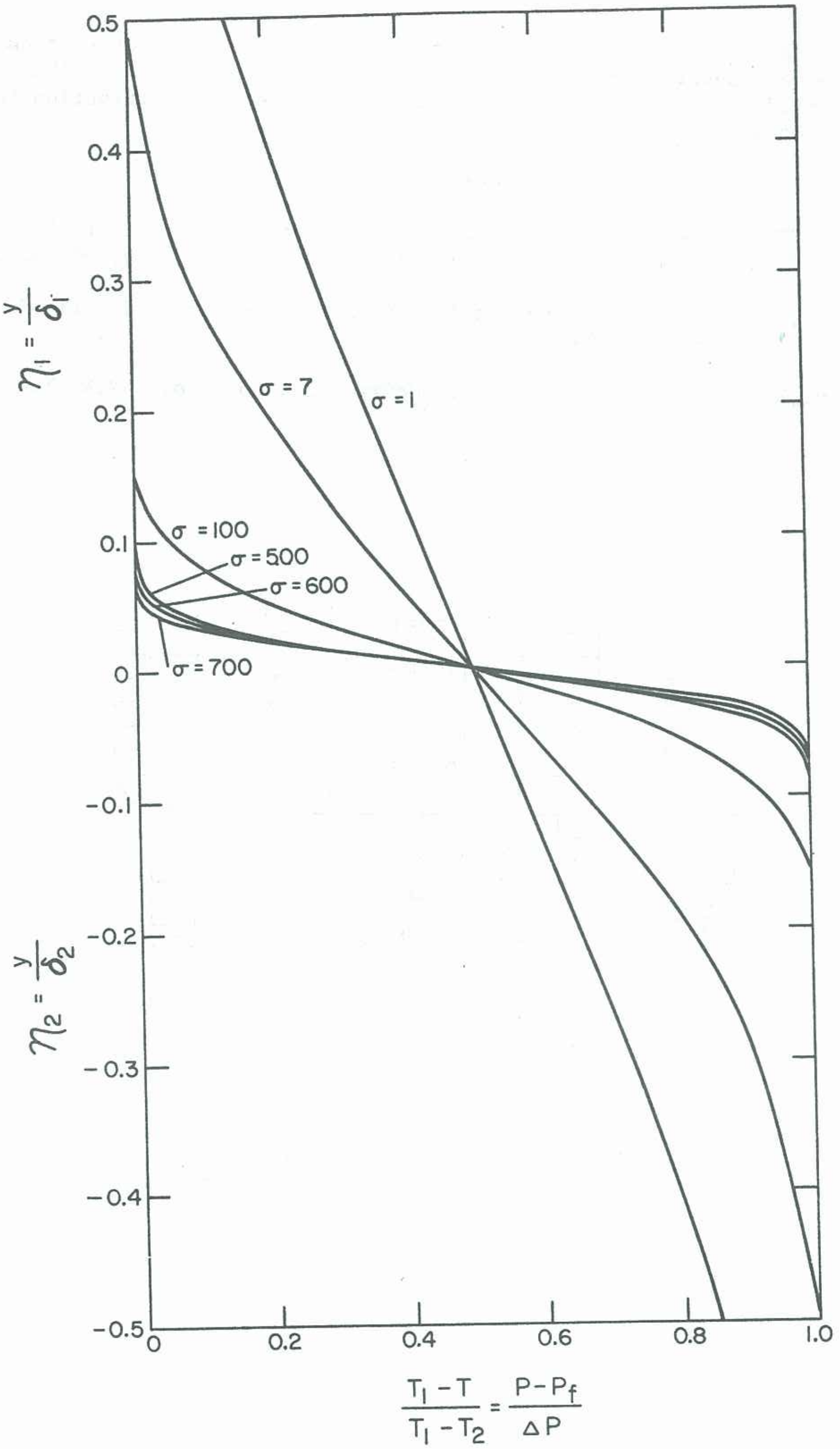

FIG. 2.—DENSITY PROFILE.

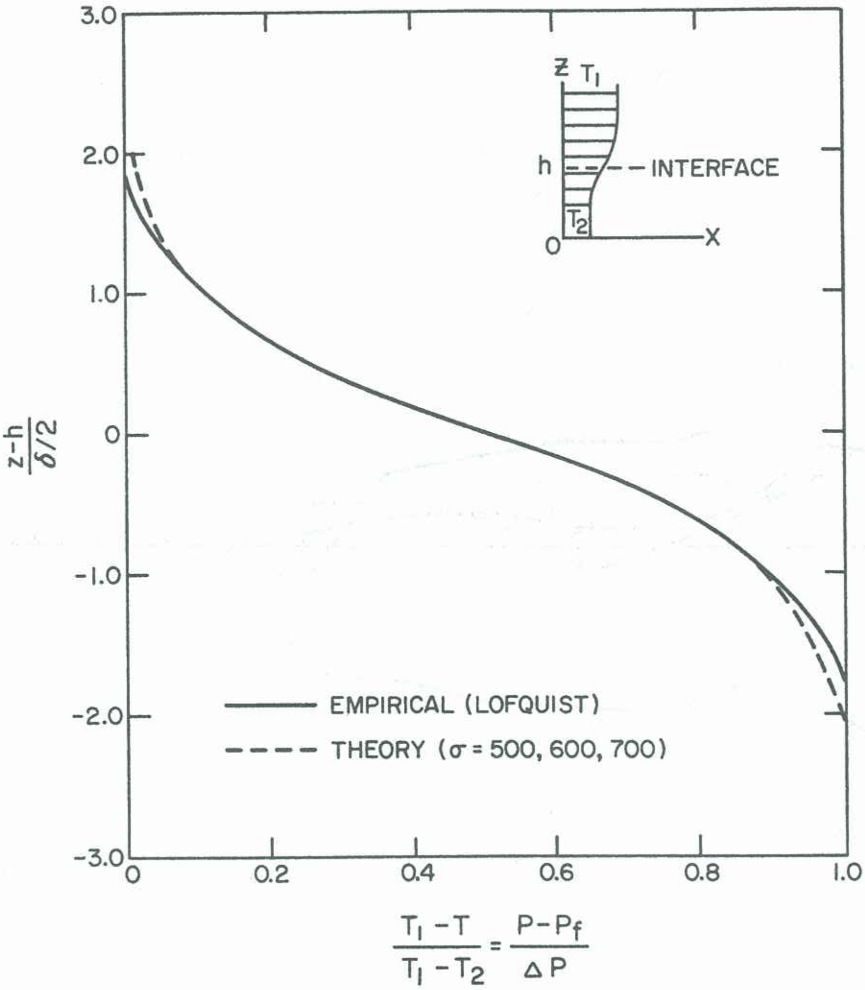

FIG. 3.—COMPARISON OF DENSITY PROFILES BETWEEN THEORY AND EXPERIMENT (LOFQUIST).

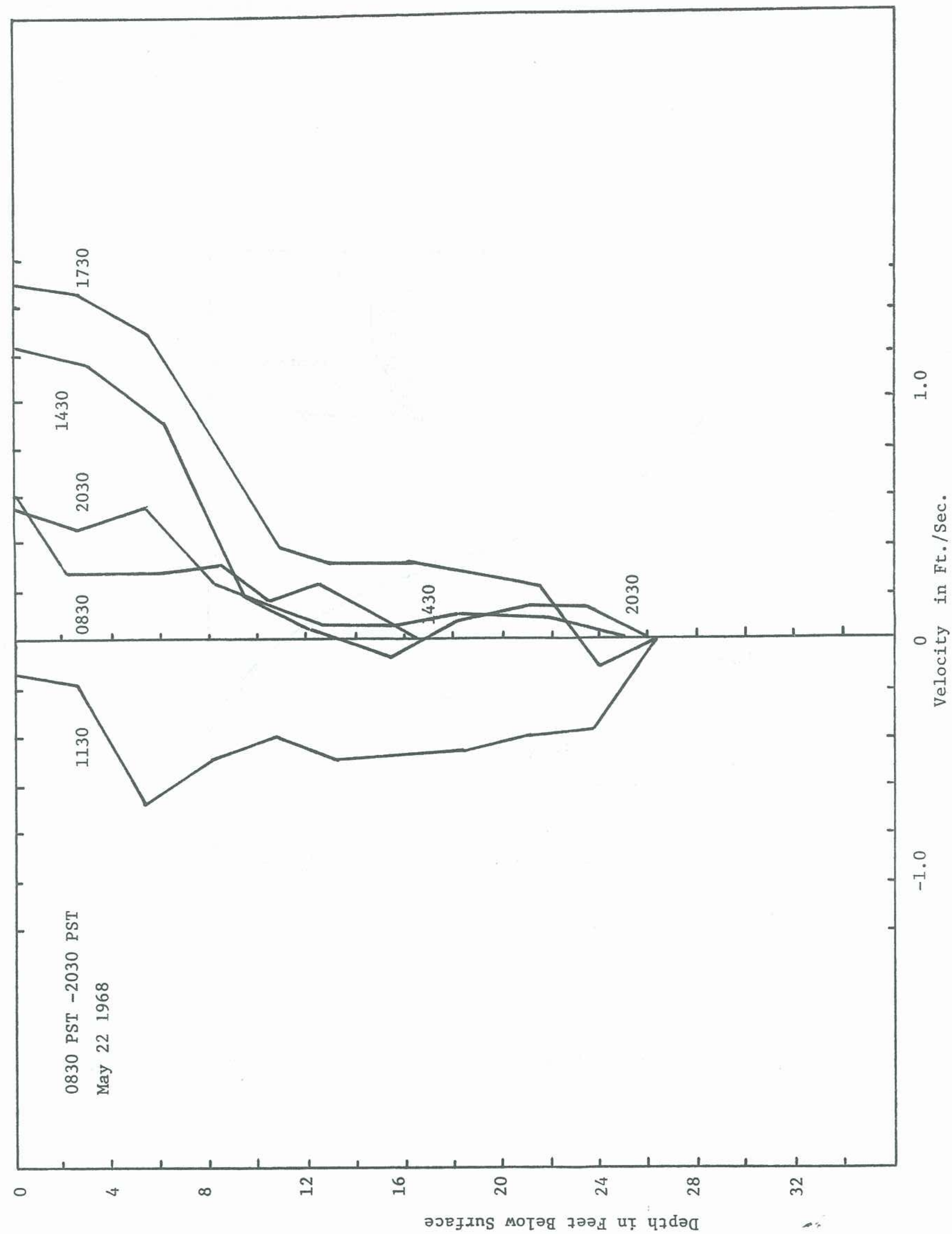

Figure 4. Velocity vs. Depth in Duwamish River (U.S.Geological Survey Data)

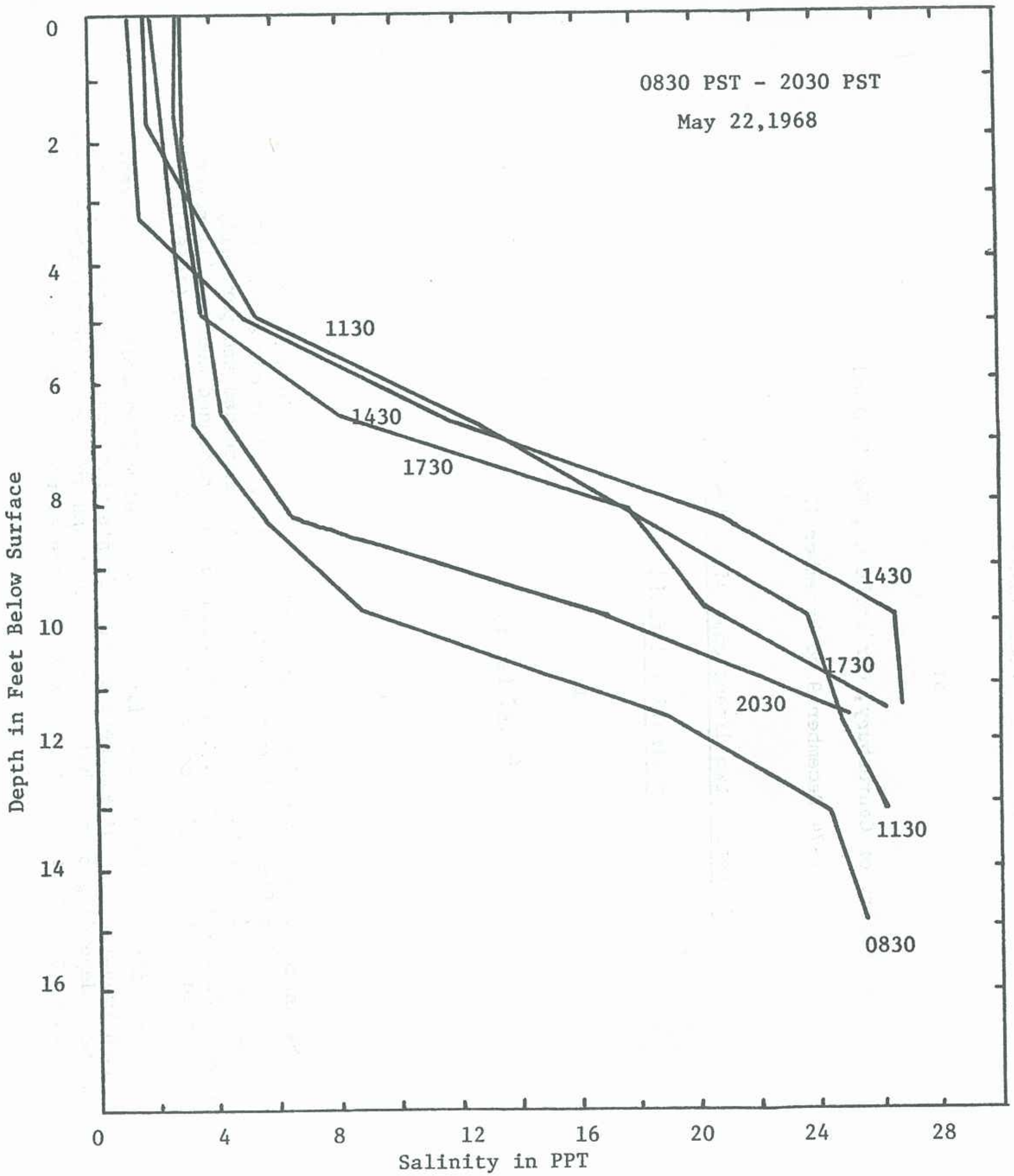

Figure 5. Salinity vs Depth in Duwamish River(U.S.Geological Survey Data)