FIFTH AUSTRALIASIAN CONFERENCE

on

HYDRAULICS AND FLUID MECHANICS

at

University of Canterbury, Christchurch, New Zealand

1974 December 9 to December 13

Nonlinear Flow Over Wavy Topography

by

F. A. Lee

SUMMARY

Flow behavior over wavy topography has been studied on the basis of a perturbation expansion procedure. The perturbed turbulent stresses are related to the perturbation strain rate in terms of an eddy viscosity coefficient. Calculation of the distances were carried out by integrating the perturbation equation of Orr-Sommerfeld kind.

It was shown that such a perturbation procedure gave a reasonable pressure and shear stress distribution over wavy topography when the Reynolds number based on boundary layer thickness is less than 10,000. In the case of flow over natural sand wave, the flow Reynolds number usually reaches the order of million. Henceforth the flow near the bed becomes highly nonlinear. By studying recent flow measurements carried out in Columbia River by Smith, A two layer flow model is proposed. A layer close to the bed will be considered to be dominated by nonlinear terms with turbulence generated by local shear next to the bed. And a layer immediately above will be considered as flow perturbed by the layer below. Turbulence within this layer derives from turbulence advected from upstream and diffused from below.

F. A. Lee, Department of Oceanography, University of Washington, Seattle, Washington, U. S. A. 98195

List of Symbols

a	wave amplitude
E	eddy viscosity coefficient applied to the gradient of the perturbations
k	wavenumber
K	an empirical constant
P	mean pressure
p	perturbation pressure
R	an empirical constant
R_e	Reynolds number
u	total velocity in x-direction
U	mean
z	distance normal to wave surface
μ	an empirical constant
ε	thickness of lower layer
χ	curvilinear coordinate tangent to wave surface
δ	boundary layer thickness
κ	Karman's constant
ϕ	streamfunction coefficient which is only dependent on z
ψ	streamfunction
$\hat{}$	denotes perturbation coefficient which is only dependent on z

Introduction

The object of the present study is to demonstrate that flow over wavy topography is highly nonlinear in natural environment. In the first part of this study, a perturbation procedure is used to formulate the flow problem. Both pressure and shear stress distribution over wavy topography will be presented. Its inadequacy will be discussed in flow over natural sand waves. In the second part, some of the recent flow measurements made by Smith (1) in Columbia River will be used as a basis to formulate a more realistic model to simulate flow over wavy topography. Preliminary discussions of the approach used to solve the resulting boundary value problem will be presented.

Description of the problem

Consider the flow over a wavy surface, periodic in x with wave length λ and with amplitude a. Detail of the problem formulation has been given elsewhere (2) and will not be repeated here. By introducing an eddy viscosity coefficient $E(z)$, the flow perturbation in terms of a stream function ϕ :

$$\phi = \phi_1 + ka\phi_2 + \cdots$$

and be formulated as follows:

First order

$$U(\phi_1'' - k^2\phi_1) - U''\phi_1$$

$$= (1+E)(ikR_e)^{-1}[\phi_1^{IV} - 2k^2\phi_1'' + k^4\phi_1] \qquad (1)$$

with boundary conditions

$$\phi_1(0) = 0$$

$$\phi_1'(0) = -aU'(0) \qquad (2)$$

Second order

$$U(\phi_2'' - 4k^2\phi_2) - U''\phi_2 + \tfrac{1}{2}(\phi_1'\phi_1'' - \phi_1\phi_1''')$$

$$= (1+E)(2ikR_e)^{-1}[\phi_2^{IV} - 8k^2\phi_2'' + 16k^4\phi_2] \qquad (3)$$

with boundary conditions

$$\phi_2(0) = 0$$

$$\phi_2'(0) = -ka^2U'(0) \qquad (4)$$

It may be noted that the second order equation is almost of the same type as the first order equation with wave number k replaced by $2k$. In addition, there is a forcing term $\frac{1}{2}(\phi_1'\phi_1'' - \phi_1\phi_1''')$ due to self-interaction of the first order solutions. The boundary condition in the second order problem is proportional to that of the first order problem by a factor ka.

The expressions for the shear stress and pressure distribution at the wavy bed are

$$\hat{\tau}_s = R_e^{-1}\{\phi''(0) + k^2\phi(0) + a\,U''(0)\}$$

$$\hat{p}_s = U_0'\,\phi(0) - U_0\,\phi'(0) - i(kR_e)^{-1}\{\phi_0''' - k^2\phi_0' + a[U_0''' - kU_0'']\} \qquad (5)$$

where

$$\tau_s = R_e^{-1}U'(0) + \hat{\tau}_s\,e^{ikx}$$

$$p_s = \hat{p}_s\,e^{ikx}$$

Method of Solution

When the coefficients of the governing equation (1) vary in the range of interest, no closed form solution exists. However, a numerical intergration is possible over the interval, by treating the boundary value problem as an initial value problem with two free parameters to be determined by the boundary conditions. The Kutta-Runge fourth order scheme has been used to integrate the differential equation. A detailed description of the technique with application to hydrodynamic stability is given by Brown and Lee (3).

In order to integrate the Orr-Sommerfeld equation, the mean profile $U(z)$ must be specified. The profile used in this study made use of the Cess eddy viscosity model,

$$E(z) = \frac{1}{2}\left\{1 + \frac{\kappa^2 R_e^2 R}{9}(2z - z^2)^2(3 - 4z + 2z^2)^2\left[1 - \exp\left(-\frac{R_e\sqrt{R}}{K}z\right)\right]^2\right\}^{\frac{1}{2}} - \frac{1}{2}$$

The velocity profile is then given by

$$U(z) = R_e R\int_0^z \frac{1 - z}{1 + E(z)}\,dz$$

Discussion of Results

For a Reynolds number R_e = 10,000 and a wave number k, amplitude a, both shear stress and pressure distribution are presented in Figure 1 and Figure 2. And the modified velocity profiles are shown in Figure 3. Two points are of interest. First, the maximum velocity modifications occur at a constant level above the topography. Second, the velocity modification by the topography is symmetric as a result of linear perturbation. Asymmetry occurs if nonlinearity has been taken into account. It is also possible when the topography is asymmetrical. Then the velocity modifications would be of several Fourier components associated with the corresponding topography Fourier components. Some aspects of such nonlinearity have been discussed in reference (2). Application of the preceding procedure to water flowing over large scale natural sand waves in rivers will be given below.

Flow over natural sand waves

In recent years, Smith (1) collected both mean flow and turbulence date over large scale natural sand waves in the Columbia River near Hood River, Oregon. The wave height was 2m and the wave length was 90m. It was found that neither the mean flow field nor the turbulent stress field could be fit by a single harmonic. It implied that the flow perturbation by the topography must be nonlinear.

In terms of the linearized model discussed in the first part, the lower boundary conditions require that

$$\phi_1(0) = 0$$

$$\phi_1'(0) = -a U_0'$$

where

$$U_0' = 0.0937/z_0$$

The velocity shear near the bed was obtained by curve fitting the measured mean velocity profile. And the roughness height based on the measured profile was

$$z_0 = 1.4\times 10^{-4}$$

normalized by the depth of the water = 15 m.

In the absence of topography, a stream function was defined as

$$\psi_0 = \int_0^z U(z)\, dz$$

In the presence of wavy topography,

$$\psi(x, z) = \psi_0 + [\phi(z) + a U e^{-kz}] e^{ikx}$$

The velocity component parallel to the bed is given by

$$u(x, z) = U(z) + [\phi'(z) + a U' e^{-kz}] e^{ikx}$$

Perturbation procedure assumes that $a U'(z)$ should be small compared to the mean velocity $U(z)$. However, next to the bed it is of the order of

$$a U'(z_0) = a \times .0937/z_0$$

With z_0 given above, and $a = 10^{-2}$, $a U'(z_0) = 6.7$

which is an order of magnitude larger than the maximum mean flow.

Within a distance of $100z_0$, from the bed, the perturbation would be too large to use the linearized scheme. The flow close to the bed undergoes nonlinear adjustment to accommodate the rapid flow variation along the topography. However, for the region far away from the bed, the flow may still be considered as being perturbed by a modified periodic topography. The governing equation remains the same as equation (1). The difficulty is associated with the boundary conditions. Instead of the original conditions (2), a slip boundary condition and no normal flow condition will be imposed:

$$\phi_1(0) = 0, \quad \phi_1'(0) = u_1^*(\varepsilon)$$

where the slip velocity $u_1^*(\varepsilon)$ will be provided through measurement along a streamline at a small distance above the bed. In general, the measured velocity must be Fourier decomposed into several Fourier components,

$$u^*(x, \varepsilon) = \text{Real Part}\,\{u_1^*(\varepsilon) e^{ikx} + u_2^*(\varepsilon) e^{aikx} + \cdots\}$$

The flow problem can then be determined by using the numerical procedure described in the first part.

For the layer near the bed, the governing equations will be nonlinear. Furthermore, the upper boundary conditions consist of variable tangential velocity and pressure. Both quantities must be provided through measurements. On the other hand, the lower boundary conditions remain the same as before. The flow problem must be numerically integrated through finite difference. Application of such a model to flow over natural sand waves is in progress. The results will be available in the near future.

Acknowledgement

The research reported herein was supported by the Oceanography section, National Science Foundation under Grant GA-26161 and under National Science Foundation Grant GV-28807.

The author would like to express his appreciation to Dr. J. D. Smith for numerous discussions during this study. This paper is contribution #M74-61, Department of Oceanography, University of Washington, Seattle.

References

McLean, S. R. and Smith, J. D., 1973: Eddy viscosity and stress field in non uniform flow. *Transactions American Geophysical Union* 54(11):1118-1119

Lee, F. A., 1972: Some nonlinear aspects of wind-wave interaction. *J. Phys. Ocean* 2(4):432-438.

Brown, R. and Lee, F., 1972: A shooting method application to the hydrodynamic stability problem. *J. Computational Physics,* 10(1):107-122.

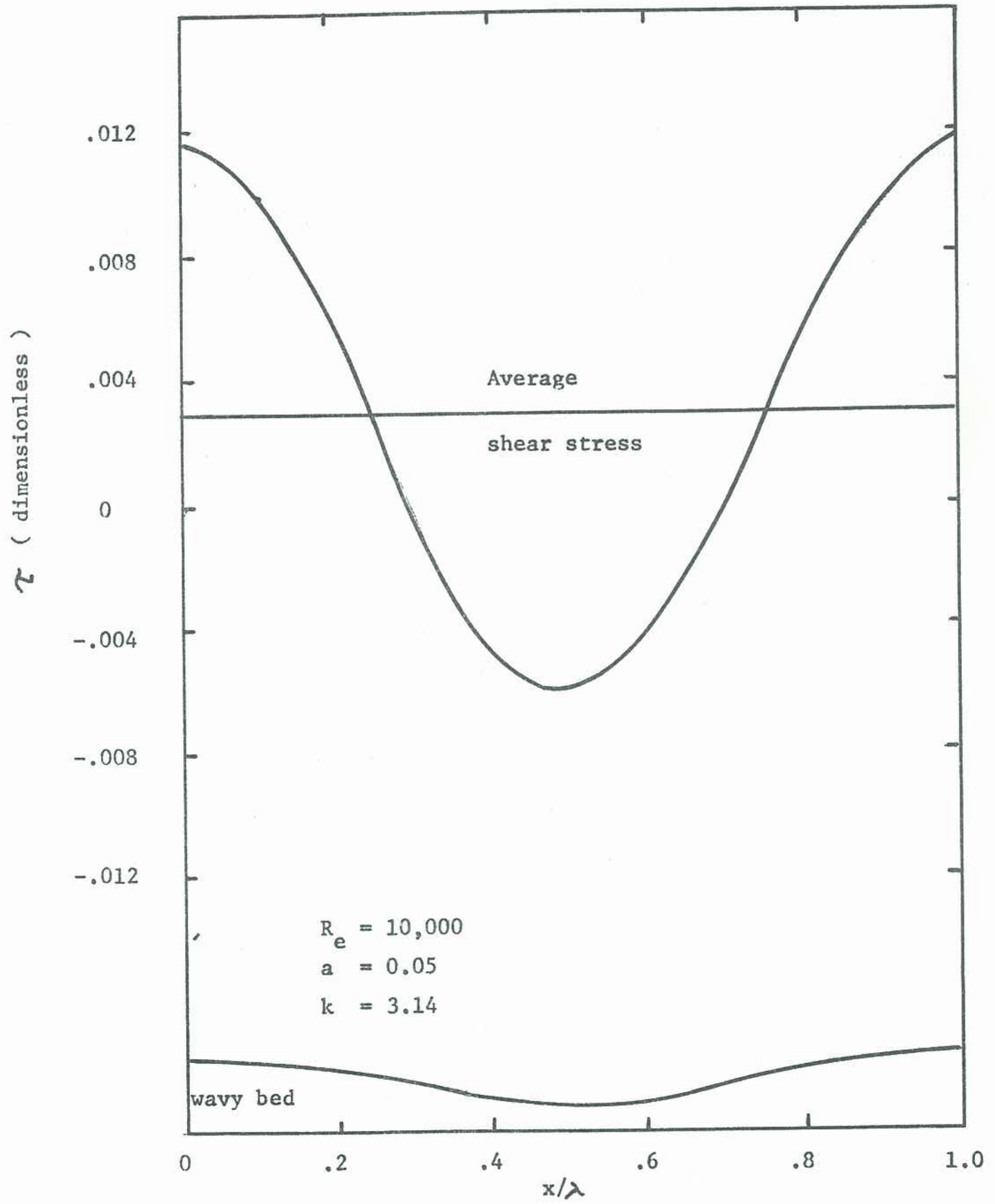

Figure 1. Shear stress distribution over a wavy bed

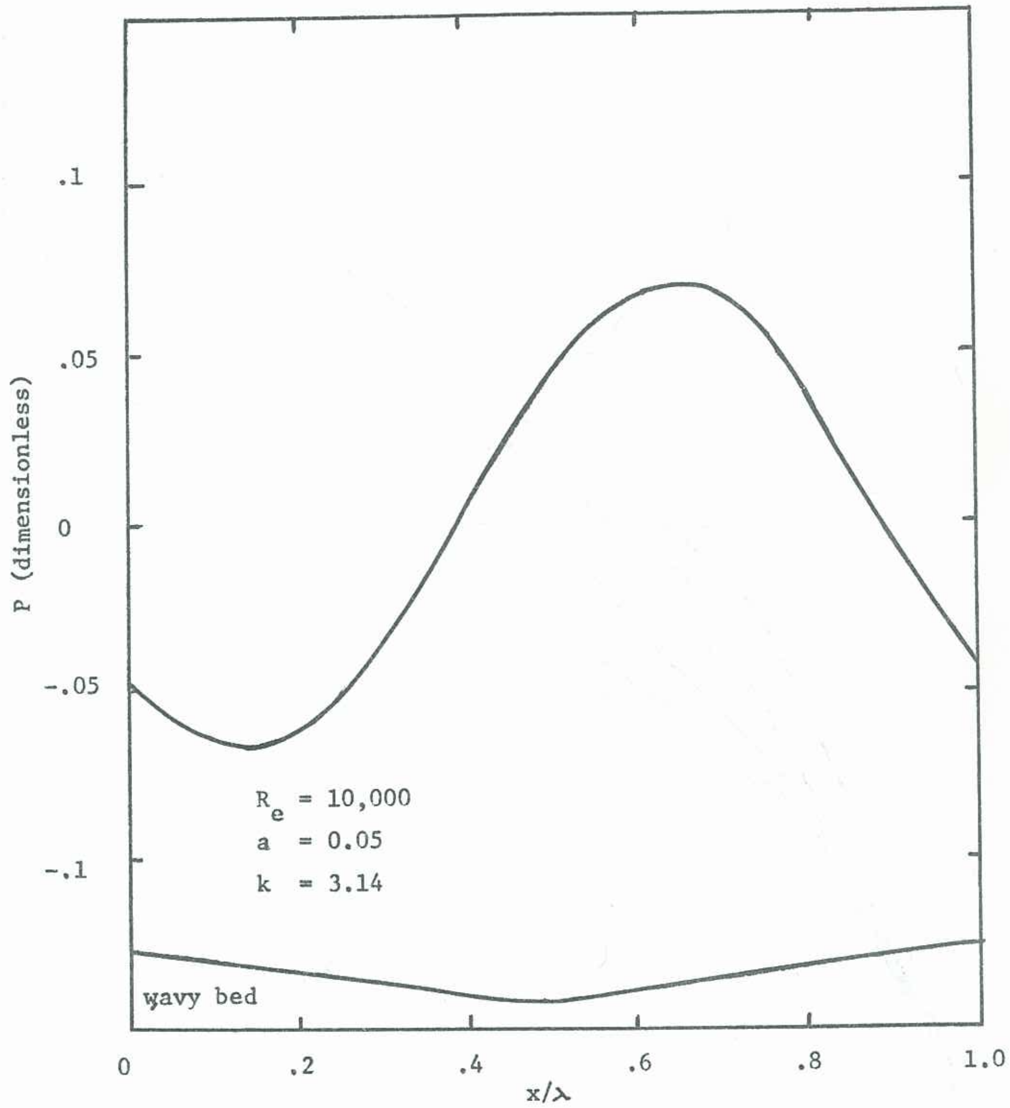

Figure 2. Pressure distribution over a wavy bed

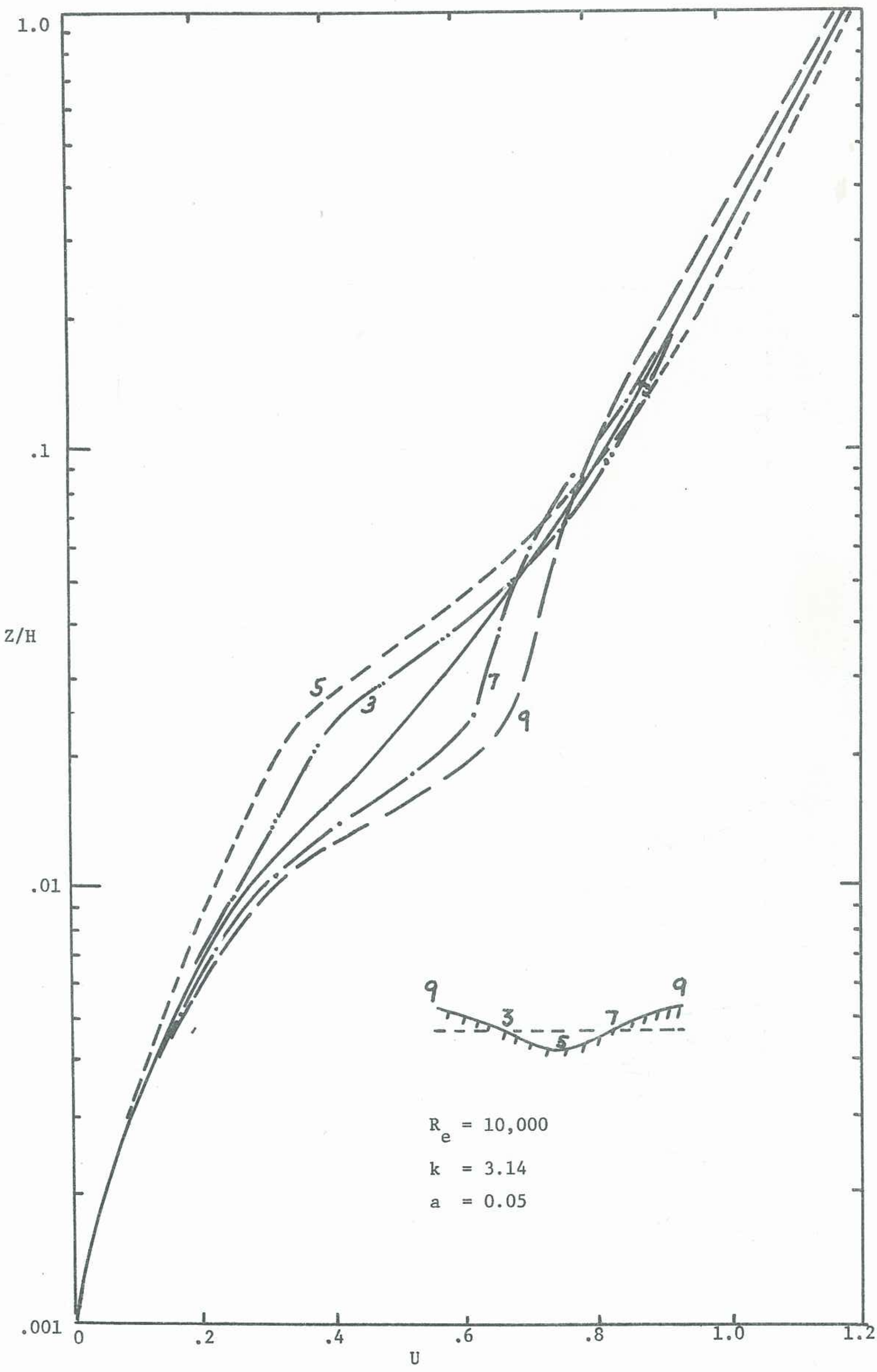

Figure 3. Flow Modification by Wavy Topography