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FREE CONVECTION IN AN INSULATED HEATING PIPE

by

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SUMMARY

Perturbation techniques are used to obtain zero and first-order solutions for streamlines and isotherms that occur in a porous insulating material surrounding a heated pipe. The solutions, which show temperature changes brought about by free convection cells in the porous insulation, are valid for relatively small Rayleigh numbers. The final results indicate that the free convection cells influence first-order temperature distributions throughout the insulation but have no influence on the total heat which is lost at the exterior surface of the insulation.

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NOTATION

A,B,C,D,E,F = constants;

c = specific heat;

H = coefficient of surface heat transfer;

$\hat{i},\hat{j}$ = horizontal and vertical unit base vectors;

k = physical or intrinsic permeability;

K_H = heat conductivity coefficient;

N = Nusselt number;

Pe = Peclet number;

R = R_1/R_2;

R_1 = outer pipe radius;

R_2 = outer insulation radius;

r = radial coordinate;

T = temperature;

T_1 = temperature at $r = R_1$;

T_∞ = ambient air temperature as $r \to \infty$;

$\bar{v}$ = velocity vector;

x,y = horizontal and vertical Cartesian coordinates;

α = $(1-T\epsilon)/(1-T\delta)$;

δ = $(\rho_\infty-\rho_1)/\rho_\infty$;

ϵ = $(\mu_\infty-\mu_1)/\mu_\infty$;

θ = angular coordinate;

μ,μ_1,μ_∞ = absolute viscosity at T,T_1 and T_∞;

ρ,ρ_1,ρ_∞ = mass density at T,T_1 and T_∞;

σ = porosity; and

ψ = streamfunction.

INTRODUCTION

Pipes conducting hot air and steam are often covered with insulation in order to reduce heat losses. These heat losses are usually calculated by assuming that heat is transferred through the insulation solely by conduction (Reference 2). Some insulations, however, are quite porous, so that heat is actually transferred through the insulation as a result of both conduction and free convection currents which exist within the porous insulation. This paper derives perturbation solutions for the streamlines and temperature distribution within such a porous insulation and attempts to evaluate the role of free convection in heat losses from an insulated pipe.

PROBLEM FORMULATION

The heated pipe surface and the exterior insulation surface have radii of R_1 and R_2. Thus, the following equations are assumed to hold in the annular region $R_1 < r < R_2$, $0 \leq \theta \leq 2\pi$:

$$\bar{v} = \frac{\rho_o}{\rho}\left(\frac{\partial\psi}{\partial y}\hat{i} - \frac{\partial\psi}{\partial x}\hat{j}\right) \qquad (1)$$

$$\bar{v} = -\frac{k}{\mu}\left(\bar{\nabla}p + \rho g\hat{j}\right) \qquad (2)$$

$$K_H\nabla^2 T = \frac{\rho c\bar{v}}{\sigma}\cdot\bar{\nabla}T \qquad (3)$$

$$\frac{\rho}{\rho_\infty} = 1 - \left(\frac{T-T_\infty}{T_1-T_\infty}\right)\delta \qquad (4)$$

$$\frac{\mu}{\mu_\infty} = 1 - \left(\frac{T-T_\infty}{T_1-T_\infty}\right)\epsilon \qquad (5)$$

The dimensionless constants $\delta \equiv (\rho_\infty-\rho_1)/\rho_\infty$ and $\epsilon \equiv (\mu_\infty-\mu_1)/\mu_\infty$ are small in many applications. For example, if the outside of the heated pipe has a temperature of $T_1 = 100^\circ C$ and the ambient air as $r \to \infty$ has a temperature of $T_\infty = 20^\circ C$, then $\delta = 0.198$ and $\epsilon = -0.185$. The following boundary conditions will be used:

$$\psi(R_1,\theta) = 0 \qquad (6)$$

$$\psi(R_2,\theta) = 0 \qquad (7)$$

$$T(R_1,\theta) = T_1 \qquad (8)$$

$$K_H\frac{\partial T(R_2,\theta)}{\partial r} + H[T(R_2,\theta) - T_\infty] = 0 \qquad (9)$$

It is convenient to formulate the problem in the following dimensionless variables:

$$(T^*,\psi^*,x^*,y^*,R^*,N,P\acute{e}) = \left(\frac{T-T_\infty}{T_1-T_\infty},\ \frac{\psi\mu_\infty}{kgR_2\rho_\infty},\ \frac{x}{R_2},\ \frac{y}{R_2},\ \frac{R_1}{R_2},\ \frac{HR_2}{K_H},\ \frac{kgcR_2\rho_\infty^2}{\sigma K_H\mu_\infty}\right) \qquad (10)$$

N is a Nusselt number and $P\acute{e}$ is a Peclet number. Substituting 10 into Eqs. 1 through 9, dropping the asterisk superscript for notational convenience and eliminating $\bar{v}, p, \rho$ and μ gives the following boundary-value problem for T^* and ψ^*:

$$\bar{\nabla}\cdot(\alpha\bar{\nabla}\psi) = -\delta\frac{\partial T}{\partial x},\ \left(\alpha \equiv \frac{1-T\epsilon}{1-T\delta}\right) \qquad (11)$$

$$\nabla^2 T = P\acute{e}\left(\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right) \qquad (12)$$

$$\psi(R,\theta) = 0 \qquad (13)$$

$$\psi(1,\theta) = 0 \qquad (14)$$

$$T(R,\theta) = 1 \qquad (15)$$

$$\frac{\partial T(1,\theta)}{\partial r} + N\,T(1,\theta) = 0 \qquad (16)$$

The region for which these equations are to be solved is $R \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

LINEARIZATION

It is assumed that the following infinite series converge:

$$\psi(r,\theta;\delta,\epsilon) = \psi_{10}(r,\theta)\,\delta + \psi_{01}(r,\theta)\,\epsilon + \ldots\ldots \tag{17}$$

$$T(r,\theta;\delta,\epsilon) = T_0(r,\theta) + T_{10}(r,\theta)\,\delta + T_{01}(r,\theta)\,\epsilon + \ldots\ldots \tag{18}$$

Substituting Eqs. 17 and 18 into Eqs. 11 through 16 and equating like powers of ϵ and δ gives the following systems of boundary-value problems:

$$\nabla^2 T_0 = 0 \tag{19}$$

$$T_0(R,\theta) = 1 \tag{20}$$

$$\frac{\partial T_0(1,\theta)}{\partial r} + N\,T_0(1,\theta) = 0 \tag{21}$$

$$\nabla^2 \psi_{10} = -\frac{\partial T_0}{\partial x} \tag{22}$$

$$\psi_{10}(R,\theta) = 0 \tag{23}$$

$$\psi_{10}(1,\theta) = 0 \tag{24}$$

$$\nabla^2 T_{10} = P\acute{e}\left(\frac{\partial\psi_{10}}{\partial y}\frac{\partial T_0}{\partial x} - \frac{\partial\psi_{10}}{\partial x}\frac{\partial T_0}{\partial x}\right) \tag{25}$$

$$T_{10}(R,\theta) = 0 \tag{26}$$

$$\frac{\partial T_{10}(1,\theta)}{\partial r} + N\,T_{10}(1,\theta) = 0 \tag{27}$$

$$\nabla^2 \psi_{01} = 0 \tag{28}$$

$$\psi_{01}(R,\theta) = 0 \tag{29}$$

$$\psi_{01}(1,\theta) = 0 \tag{30}$$

$$\nabla^2 T_{10} = P\acute{e}\left(\frac{\partial\psi_{01}}{\partial y}\frac{\partial T_0}{\partial x} - \frac{\partial\psi_{01}}{\partial x}\frac{\partial T_0}{\partial y}\right) \tag{31}$$

$$T_{01}(R,\theta) = 0 \tag{32}$$

$$\frac{\partial T_{01}(1,\theta)}{\partial r} + N\,T_{01}(1,\theta) = 0 \tag{33}$$

The solutions of Eqs. 19 through 33 can be shown to be unique since $N \geqq 0$. Furthermore, it should be pointed out that $P\acute{e}$ does not appear in the problem for T_0. This is an indication that the infinite series 17 and 18 will probably cease to converge as $P\acute{e}$ becomes large.

SOLUTION

The unique solutions for ψ_{01} and T_{01} are easily seen to be $\psi_{01} = T_{01} = 0$. Hence, viscosity variations exert only second order effects upon the flow pattern and temperature distribution. The solution for $T_0(r)$ is easily found to be

$$T_0(r) = \frac{1 - N\ln r}{1 - N\ln R} \tag{34}$$

A particular solution of Eq. 22 is $-\frac{1}{2}xT_0(r)$. Hence, one can set set $\psi_{10} = \varphi - \frac{1}{2}xT_0$ and solve a harmonic boundary-value problem for φ. The final solution for ψ_{10} is found in this way to be

$$\psi_{10}(r,\theta) = \left[A\,r + \frac{B}{r} - \tfrac{1}{2}\,r\,T_0(r)\right]\cos\theta \tag{35}$$

in which the constants A and B are defined as

$$A \equiv \frac{1-R^2 + N R^2 \ln R}{2(1-R^2)\ (1-N \ln R)} \tag{36}$$

$$B \equiv \frac{-\ N R^2 \ln R}{2(1-R^2)\ (1-N \ln R)} \tag{37}$$

The solution for T_{10} can be found by defining a function, Φ, in the following way:

$$T_{10} = P\acute{e}\ (\Phi - \tfrac{1}{2}\varphi^*\ T_0) \tag{38}$$

The function $\varphi^* \equiv (Ar - \frac{B}{r}) \sin\theta$ is the harmonic conjugate of the function φ, which was found when solving for $\psi_{10}(r,\theta)$. The function Φ is then found to satisfy the following Poisson's equation:

$$\nabla^2\Phi = \tfrac{1}{2}\ T_0 \frac{\partial T_0}{\partial y} = -\frac{(1-N \ln r)}{(1-N \ln R)^2}\left(\frac{N}{2r}\right) \sin\theta \tag{39}$$

Eq. 39 and its appropriate boundary conditions, obtained from Eqs. 26, 27 and 38, can be solved by separating variables. The result, when substituted into Eq. 38, yields

$$T_{10}(r,\theta) = P\acute{e}\left[Cr + \frac{D}{r} + \frac{N\ r \ln r\ (N \ln r-2-N)}{8(1-N \ln R)^2} - \frac{T_0(r)}{2}\ (Ar - \frac{B}{r})\right] \sin\theta \tag{40}$$

The constants C and D are defined as

$$C \equiv \frac{(1-N)E + \frac{F}{R}}{\frac{(1+N)}{R} + (1-N)R} \tag{41}$$

$$D \equiv \frac{(1+N)E - RF}{\frac{(1+N)}{R} + (1-N)R} \tag{42}$$

in which the constants E and F are given by

$$E \equiv \tfrac{1}{2}(AR - \frac{B}{R}) - \frac{N\ R \ln R\ (N \ln R-2-N)}{8(1-N \ln R)^2} \tag{43}$$

$$F \equiv \frac{(A+B)}{2(1-N \ln R)} + \frac{N(2+N)}{8(1-N \ln R)^2} \tag{44}$$

DISCUSSION

A plot of isotherms and streamlines calculated for $N=1.0$, $R=0.3$ and $P\acute{e}\ \delta=10.0$ is shown in FIG. 1. This plot shows that two convection cells appear on opposite sides of the pipe and the effect of these cells is to deflect isotherms in the direction of the velocity vector. The net result is that the top of the pipe has a higher temperature than the bottom. Interesting enough, this rearrangement of the temperature distribution by the convection cells does not change the total amount of heat which is lost from the insulation surface. This is because the heat flux contributed by T_{10} on $r=1$ is proportional to $\sin\theta$, so that the integral of this heat flux vanishes and one is left with only that portion of the heat flux which is contributed by the zero-order term, $T_0(r)$. This is the heat flux that is calculated by assuming that heat is transferred by conduction only.

Finally, it should be pointed out that the solution given herein is only valid for relatively small values of δ, ε and Pe. In particular, the term $T_{10}(r,\theta)\delta$ in Eq. 18 consists of the product of a term whose magnitude is usually less than unity with the constant $P\acute{e}\delta$. This implies that Eq. 18 will give a good approximation for smaller values of the number $P\acute{e}\delta$, which is exactly equal to a Rayleigh number that has been defined and used in free convection flows of groundwater (for example, in references 1, 3 and 4). In practice, it will be found that the particular range for which Eq. 18 converges rapidly will depend upon the Nusselt number, N. Hence, it will be necessary to plot these solutions in the way shown in FIG. 3 and then discard as unreliable any solutions whose first-order isotherms show a marked variation from the zero-order isotherms.

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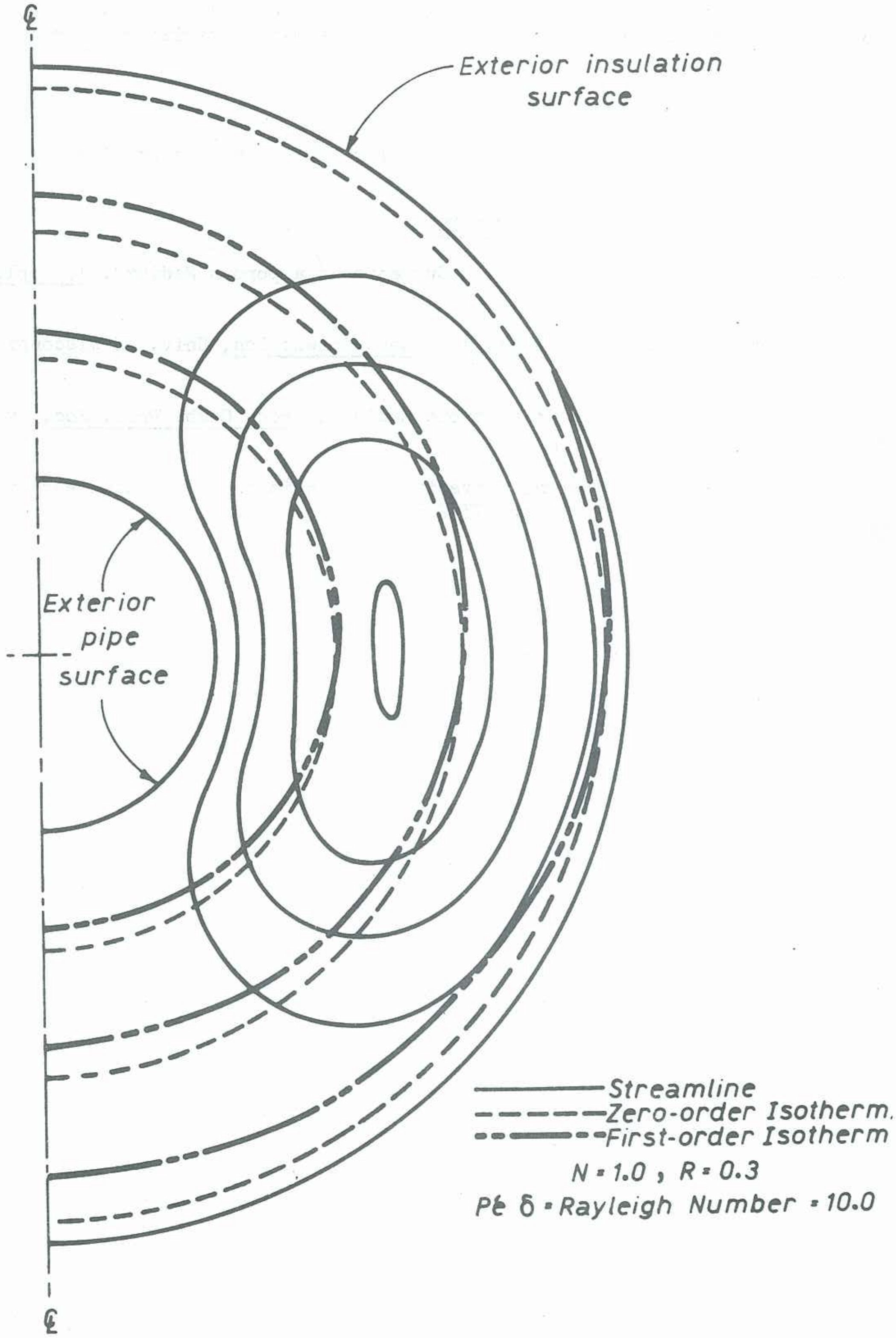

Fig. 1 – STREAMLINES AND ISOTHERMS FOR FREE CONVECTION IN AN INSULATED HEATING PIPE