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Unsteady, two dimensional sink flow in a stratified fluid

by

* C.B. Fandry

SUMMARY

The unsteady two-dimensional flow induced by a sink in a linearly stratified, viscous diffusive fluid is investigated. Analytical solutions are found for the linear equations governing the motion in both unbounded and partially bounded mediums. It is shown that the flow in an unbounded medium will never reach a steady state, which contradicts the solution found by Koh (1966b).

A solution for the unsteady flow in a parallel walled duct is also given. The main result is the development of flow fronts which move away from the sink with speed depending on the height of the duct and the degree of stratification. In the case of a viscous fluid these fronts are eventually attenuated at infinity.

* Department of Mathematics, Clayton, Victoria, Australia.

Introduction: The general problem of the selective withdrawal of fluid from a stratified reservoir has been investigated by numerous authors. Some have developed models in an unbounded fluid domain (Koh (6), (7)), and others have considered flows in partially or fully bounded domains (Debler (2), Kao (4), (5), Yih (9), Imberger (3)), the prime objective being to determine the thickness of the withdrawal layer. Difficulties associated with upstream boundary conditions of the vertically bounded models were circumvented by Imberger (3) by considering two separate regions of the flow, one near the sink where inertia forces were important and the other far from the sink where the flow was governed by a balance of buoyancy and viscous forces only. The upstream boundary condition of the inertial region was then the near sink limit of the non-inertial region.

The steady solution of Koh (7) in an unboundedrregion is fundamentally incorrect. He investigated the problem of the effect of viscosity in the stratified flow towards a sink using the linear, boundary layer equations with the conservation of volume flux across a vertical section as a boundary condition. However, as pointed out by Imberger (3), there is an inconsistency between the momentum flux or flow force (Brooke-Benjamin (1), Morton (8) given by $\int_{-\infty}^{\infty} p\,dy$ (p is the perturbation pressure and y the vertical co-ordinate), calculated from his solution and the integral form of the original momentum equations. His equations admit a similarity solution such that the flow force is given by

$$\int_{-\infty}^{\infty} p\,dy = J \sim x^{1/3} \int_{-\infty}^{\infty} P(\zeta)\,d\zeta \qquad (1)$$

where x is the horizontal co-ordinate, P the similarity pressure function and $\zeta = y/x^{1/3}$ the similarity variable. The integral form of the x-momentum equation implies that J is independent of x, and this would only be true if $\int_{-\infty}^{\infty} P(\zeta)\,d\zeta = 0$. Calculation of this quantity from Koh's solution shows that $\int_{-\infty}^{\infty} P(\zeta)\,d\zeta = \lim_{\zeta \to \infty} \zeta P(\zeta)$, and hence the condition $\lim_{\zeta \to \infty} \zeta P(\zeta) = 0$ is essential for a consistant solution.

This fundamental difficulty in the steady flow can be overcome if instead one specifies as a boundary condition the conservation of the flow force; i.e. $\int_{-\infty}^{\infty} p(x,y)\,dy$ = constant. The integral form of the continuity equation requires that the horizontal volume flux is some constant, Q; i.e. $\int_{-\infty}^{\infty} u(x,y)dy = Q$. From the solution it is shown that

$$Q = x^{-1/3} \int_{-\infty}^{\infty} U(\zeta)\,d\zeta \ ,$$

and that in fact $\int_{-\infty}^{\infty} U(\zeta)\,d\zeta = 0$, implying, of course, that Q = 0. The problem is then wholly consistent with no more restrictions on any of the perturbation variables.

In order to overcome the difficulties associated with steady two dimensional flows in unbounded domains it is necessary to include time dependence. Using order of magnitude arguments it is shown that the time dependent problem of the viscous, stratified flow towards a sink admits physically realistic and mathematically consistent solutions for both the cases of prescribed volume flux or prescribed flow force as functions of time. The conservation of volume flux requires that asymptotically for large time t, the flow force grow at the rate $t^{1/2}$ and that the conservation of flow force implies a volume flux decrease at the rate $t^{-1/2}$.

The question of consistency between solutions and integral constraints does not arise in the problem of flow in a bounded region. In the case of a vertically bounded flow it is shown that the action of the sink sends out a series of internal gravity waves moving out to horizontal infinity with speeds $(\varepsilon g)^{+1/2}/(n\pi/h)$ where h is the vertical length scale, and $(\varepsilon g)^{1/2}$ the Brunt Vaisala frequency. If the Schmidt number $\nu/\kappa = 1$, where ν is the kinematic viscosity and κ the thermal diffusion, these waves are attenuated horizontally with attenuation constant $(n\pi/h)^3\nu/(\varepsilon g)^{1/2}$. At time t, flow fronts exist at positions $x = t(\varepsilon g)^{1/2}/(n\pi/h)$, with the primary wave (n = 1), which exhibits the most concentrated front, dominating the flow pattern.

2. Governing Equations The non-dimensional, linearized boundary layer equations governing the unsteady motion in an initially linearly stratified, viscous, diffusive fluid are:-

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \sigma^{1/2}\frac{\partial^2 u}{\partial y^2} \ , \qquad (1a)$$

$$0 = -\frac{\partial p}{\partial y} - s \ , \qquad (1b)$$

$$\frac{\partial s}{\partial t} - v = \sigma^{1/2} \frac{\partial^2 s}{\partial y^2} \quad , \qquad (1c)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \qquad (1d)$$

where x and y are the horizontal and vertical co-ordinates respectively and u and v the corresponding velocity components. p and s are the pressure and density perturbation due to the motion and the Schmidt number is defined by $\sigma = {}^{\nu}/\kappa$ where ν is the kinematic viscosity and κ the thermal diffusion of the fluid. Assumptions made in the derivation of these equations are well outlined in (3) and (7). It is important to note that the vertical coordinate y is scaled according to the stratification length $\varepsilon^{-\frac{1}{3}}$, where ε is the unperturbed linear density gradient, and the horizontal co-ordinate x according to $\left(\frac{\varepsilon g}{\nu\kappa}\right)^{1/2}/\varepsilon^3$. These equations describe the motion far from the source or sink where inertia forces are negligible.

3. Boundary Conditions and Order of Magnitude Analysis

3.1 Steady Flow: Koh (7) solved the steady version of equations (1) with the integral constraint

$$\int_{-\infty}^{\infty} u(x,y)\, dy = Q \quad , \qquad (2)$$

where Q is the constant, non-zero volume flux across each vertical section at x. The asymptotic solution of equations (1) and (2) for large x and y can be determined by an order of magnitude analysis. If X, Y, U, Y, P, S represent the orders of magnitude of the corresponding lower case variables we can write equations (1) and (2) in the form $\frac{P}{Y} \sim S$, $V \sim {}^{S}/Y^2$, $\frac{U}{X} \sim \frac{V}{Y}$, $UY \sim Q$ where for simplicity we have taken $\sigma = 1$. It is then not difficult to show that these order of magnitude equations imply $U \sim QX^{-1/3}$, $V \sim QX^{-1}$, $P \sim Q$, $S \sim QX^{-1/3}$. In other words equations (1) and (2) admit a similarity solution in which the similarity variable is $\eta = {}^{y}/x^{1/3}$ and the dynamical variables are of the form

$$u(x,y) - x^{-1/3}\, a(\eta) \qquad (3a)$$

$$v(x,y) = x^{-1}\, b(\eta) \qquad (3b)$$

$$p(x,y) = c(\eta) \qquad (3c)$$

$$s(x,y) = x^{-1/3}\, d(\eta) \; . \qquad (3d)$$

Equation (3c) implies that the flow force is given by

$$\int_{-\infty}^{\infty} p(x,y)dy = x^{1/3} \int_{-\infty}^{\infty} c(\eta)\, d\eta \; ,$$

which would contradict the integral form of equation (1a) (assuming $\frac{\partial u}{\partial y} \to 0$ as $y \to \pm\infty$), which is ${}^{\partial}/\partial x \int_{-\infty}^{\infty} p(x,y)\, dy = 0$ unless $\int_{\infty}^{\infty} c(\eta)\, d\eta = 0$. Thus in addition to the integral constraint (2), a physically and mathematically consistent solution would require the restriction of zero flow force across each section; i.e. $\int_{-\infty}^{\infty} p(x,y)\, dy = 0$. Koh (7) makes no mention of this difficulty, and his solution, in general, fails to satisfy the zero flow force condition, and therefore violates the original momentum equations.

In fact the only consistent boundary condition one can impose on the steady problem is the constancy of the flow force;

$$\text{i.e.} \int_{-\infty}^{\infty} p(x,y)\, dy = F \; . \qquad (4)$$

The steady equations then imply the following orders of magnitude $U \sim FX^{-2/3}$, $V \sim FX^{-4/3}$, $P \sim FX^{-1/3}$, $S \sim FX^{2/3}$, . The horizontal velocity similarity function now has the form $u(x,y) = x^{2/3}\, a(\eta)$, and the volume flux across each vertical section x is given by

$$Q = \int_{-\infty}^{\infty} u(x,y)\, dy = x^{-1/3} \int_{\infty}^{\infty} a(\eta)\, d\eta \; ,$$

and from the solution given by

$$u(x,y) = Fx^{-2/3} \int_{0}^{\infty} k\, e^{-k^3} \cos k\eta\, dk$$

it is easily shown that Q = 0. This is consistent with the integral form of the continuity equation , $\partial/\partial x \int_{-\infty}^{\infty} u(x,y)dy = 0.$

This is the only physically realistic steady, two dimensional solution. It consists of a flow in which through every vertical section, the flow force is constant and the volume flux zero.

3.2 Unsteady Flow : Two time-dependent initial value problems are solved. Firstly one in which the volume flux is a specified function of time and secondly one in which the flow force is a specified function of time.

Dimensional analysis of the time dependent equations in which the volume flux condition is given as

$$\int_{-\infty}^{\infty} u(x,y,t)\, dy = Q(t) \operatorname{sgn} x \qquad (5)$$

implies the following relationships $Y \sim X^{1/3}$, $T \sim X^{2/3}$, $U \sim QT^{-1/2}$, $V \sim QT^{-3/2}$, $P \sim Q$, $S \sim QT^{-1/2}$. This solution determines a flow force having order of magnitude $QT^{1/2}$. This does not violate the integral statement of the x-momentum equation which has the form

$$\int_{-\infty}^{\infty} p(x,y,t)\, dy = x \frac{dQ}{dt} \sim QT^{1/2} .$$

The volume flux condition can be incorporated into the continuity equation (1d) using the Dirac delta function in the following manner

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2\, Q(t)\, \delta(x)\, \delta(y) . \qquad (6)$$

Solution of equations (1a, b, c) and (6) with $\sigma = 1$ is carried out by taking a Laplace transform in t and Fourier transforms in x and y. The transformed variables then have the form

$$(U, V, P, S,) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\bar{u}, \bar{v}, \bar{p}, \bar{s}) e^{ik_1x+ik_2y}\, dx\, dy$$

where

$$(\bar{u}, \bar{v}, \bar{p}, \bar{s}) = \int_0^{\infty} (u, v, p, s)\, e^{-qt}\, dt.$$

Assuming that u, v, p, s $\to 0$ as $x^2+y^2 \to \infty$ and that $u = s = 0$ at $t = 0$, it is not difficult to show that

$$U(k_1, k_2, q) = - \frac{2ik_1\, \bar{Q}(q)}{\Delta(k_1,k_2)} \qquad (7a)$$

$$V(k_1, k_2, q) = \frac{2ik_2 (q+k_2^2)^2 \bar{Q}(q)}{\Delta(k_1,k_2)} \qquad (7b)$$

$$P(k_1, k_2, q) = \frac{(q+k_2^2)\bar{Q}(q)}{\Delta(k_1,k_2)} \qquad (7c)$$

$$S(k_1, k_2, q) = \frac{2ik_2 (q+k_2^2)\bar{Q}(q)}{\Delta(k_1,k_2)} \qquad (7d)$$

where

$$\bar{Q}(q) = \int_0^{\infty} Q(t)\, e^{-qt}\, dt \qquad (8)$$

and

$$\Delta(k_1,k_2) = (q+k_2^2)^2\, k_2^2 + k_1^2 .$$

The singularity of the transforms at $k_1 = k_2 = 0$ is due to the discontinuity of the volume flux across the x-section at the origin.

By inverting the transforms in equation (7a), the horizontal velocity component becomes

$$u(x,y,t) = - 2 \int_0^{t/x} e^{-xk_2^3}\, Q(t-k_2x) \cos k_2y\, dk_2. \qquad (9)$$

Then for $x > 0$, the substitution $k = k_2x^{1/3}$, $\tau = t/x^{2/3}$ and $\zeta = y/x^{1/3}$ in equation (11) leads to

$$u(x,y,t) = -2x^{-1/3} \int_0^{\tau} e^{-k^3} Q\left(x^{2/3}(\tau-k)\right) \cos(k\zeta)\, dk \qquad (10)$$

Considering the special case of a constant volume flux with $Q(t) = - H(t)$, (10) becomes

$$u(x,y,t) = 2x^{-1/3} \int_0^{\tau} e^{-k^3} \cos(k\zeta)\, dk \qquad (11)$$

Similar expressions can be found for the other dynamical variables.

From the solution in equation (11) it is now clear why no steady solution exists for two dimensional sink flows in an unbounded medium. Since the time variable t is incorporated into a similarity variable τ as $t/x^{2/3}$, τ can be made to remain constant as $t \to \infty$ if x is chosen large enough. Hence, for large values of time, t, the flow is still unsteady at large distances from the sink. In the second problem, the flow force condition

$$\int_{-\infty}^{\infty} p(x,y,t)\, dy = F(t) \operatorname{sgn} x \qquad (12)$$

leads to the following order of magnitude relationships $Y \sim X^{1/3}$, $T \sim X^{2/3}$, $U \sim FT^{-2}$, $P \sim FT^{-1/2}$ $S \sim FT^{-1}$ and the volume flux has order of magnitude $FT^{-1/2}$. Incorporating (12) into the x-momentum equation as a body force like:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + 2F(t)\,\delta(x)\,\delta(y) , \qquad (13)$$

solution of equations (1b), (1c), (1d) and (13) for the case $F(t) = H(t)$ follows analogous lines to the above solution for the first problem, and the horizontal velocity component can be written as :-

$$u(x,y,t) = 2x^{-2/3} \int_0^{\tau} k\, e^{-k^3} \cos(k\zeta)\, dk. \qquad (14)$$

The horizontal velocity profiles given by (14) are plotted in figure 2. Comparison of these flow profiles with thoseof the previous case given by equation (11) in figure 1 reveal one basic similarity. The withdrawal layer thickness decreases as τ increases, and due to the action of viscosity, reaches a fixed value equal to $2.1\, x^{1/3}$ and $3.5\, x^{1/3}$ in the constant flow force problem and constant (non-zero) volume flux problem respectively after about $\tau = 1.0$. The main difference between the two sets of velocity profiles is that the region of forward flow in the constant volume flux problem is much stronger, and this is accompanied by a correspondingly weaker reverse flow. The vertical density gradient is much more effective in inhibiting vertical motion in this case, and the flow is confined to a more concentrated withdrawal layer.

4. Unsteady Flow in Partially Bounded Domains

In this section we shall consider the flow induced by a sink at the origin in a fluid domain bounded above and below by rigid walls at $y = \pm h$. Equations (1) govern the unsteady motion, and in terms of the stream function ψ where $u = -\frac{\partial \psi}{\partial x}$, $v = \frac{\partial \psi}{\partial y}$, the boundary conditions appropriate for a sink withdrawing a constant volume Q are

$$\left.\begin{aligned}
\psi &= Q/2\, H(t) \operatorname{sgn} x , \; y = h \\
\psi &= -Q/2\, H(t) \operatorname{sgn} x , \; y = -h \\
\psi &= Q/2\, H(t) \operatorname{sgn}(y/h), \; x = 0 \\
\frac{\partial^2 \psi}{\partial y^2} &= 0 , \; \frac{\partial^4 \psi}{\partial y^4} = 0 , \; t = 0
\end{aligned}\right\} \qquad (15)$$

where H(t) is the Heaviside step function.

Elimination of the other variables in favour of ψ from the dimensional form of equation (1) yields

$$\nu\kappa \frac{\partial^6 \psi}{\partial y^6} + \frac{\partial^4 \psi}{\partial t^2 \partial y^2} - (\nu+\kappa) \frac{\partial^5 \psi}{\partial t \partial y^4} + \varepsilon g \frac{\partial^2 \psi}{\partial x^2} = 0 \qquad (16)$$

By taking a Laplace Transform in t and using Fourier analysis on equation (16), the solution satisfying boundary conditions (15) for the simplified case $\nu/\kappa = 1$ is given by

$$\psi(x,y,t) = \frac{Q}{\pi} \sum_{n=1}^{\infty} \frac{1}{\pi} \exp\left(- \frac{n^3\pi^3 x}{h^3} \nu(\varepsilon g)^{-1/2}\right) \sin\left(\frac{n\pi y}{h}\right) H\left(t - \frac{n\pi x}{h} (\varepsilon g)^{-1/2}\right)$$

$$+ {}^{Q}/_{2} \frac{y}{h} H(t) \;\ldots\ldots\ldots \qquad (17)$$

Except for the time dependent terms, this solution is identical with that found by Imberger (3). The solution exhibits sharp horizontal gradients or flow fronts at positions $x_n = \frac{t(\varepsilon g)^{1/2}}{n\pi/h}$, which move out to infinity with speed $\frac{(\varepsilon g)^{1/2}}{(n\pi/h)}$. The streamlines given by equation (17) are plotted for various values of t in figure 3. Values of t were chosen much larger than the Brunt Väisälä period $(\varepsilon g)^{-1/2}$ when the governing equations are valid.

The flow patterns depicted in figures 3(a), (b) clearly show the sharp front occurring at $x_1 = \frac{t(\varepsilon g)^{1/2}}{\pi/h}$ and a much weaker front at $x_2 = {}^{x_1}/2$. These fronts correspond to a series of gravity waves propagating out to infinity, but for all $x \gtrsim X$, where $X \gg \left(\frac{n^3\pi^3}{h^3} \nu(\varepsilon g)^{-1/2}\right)^{-1}$, the amplitudes of these waves, and hence the intensity of the fronts, are negligible. This means that for times t greater than $\frac{n\pi X}{h} (\varepsilon g)^{-1/2}$, fronts will no longer be distinguishable and the flow will be essentially steady from then on. Figure 3(c) shows the flow after a long time, but not long enough for it to have reached a steady state. Imberger's (3) outer solution depicts the steady state flow.

References

(1) Brooke Benjamin, T. 1962 Theory of the vortex breakdown phenomenon. J. Fluid Mech., 14, 593.

(2) Debler, W.R. 1959 Stratified flow into a line sink. Proc. Amer. Soc. Civil Engrs., 85, No. EM 3, 51.

(3) Imberger, J. 1972 Two-dimensional sink flow of a stratified fluid contained in a duct. J. Fluid Mech., 53, 329.

(4) Kao, T.W. 1965 A free-streamline solution for stratified flow into a line sink. J. Fluid Mech., 21, 535.

(5) Kao, T.W. 1970 Free-streamline theory for inviscid stratified flow into a line sink. Phys. Fluids, 13, 558.

(6) Koh, R.C.Y. 1966a Unsteady stratified flow into a sink. J. Hydraulic Res., 4, 21.

(7) Koh, R.C.Y. 1966b Viscous stratified flow towards a sink. J. Fluid Mech., 24, 555.

(8) Morton, B.R. 1969 The strength of vortex and swirling core flows. J. Fluid Mech., 38, 315.

(9) Yih, C.S. 1965 Dynamics of Nonhomogeneous Fluids. Macmillan.

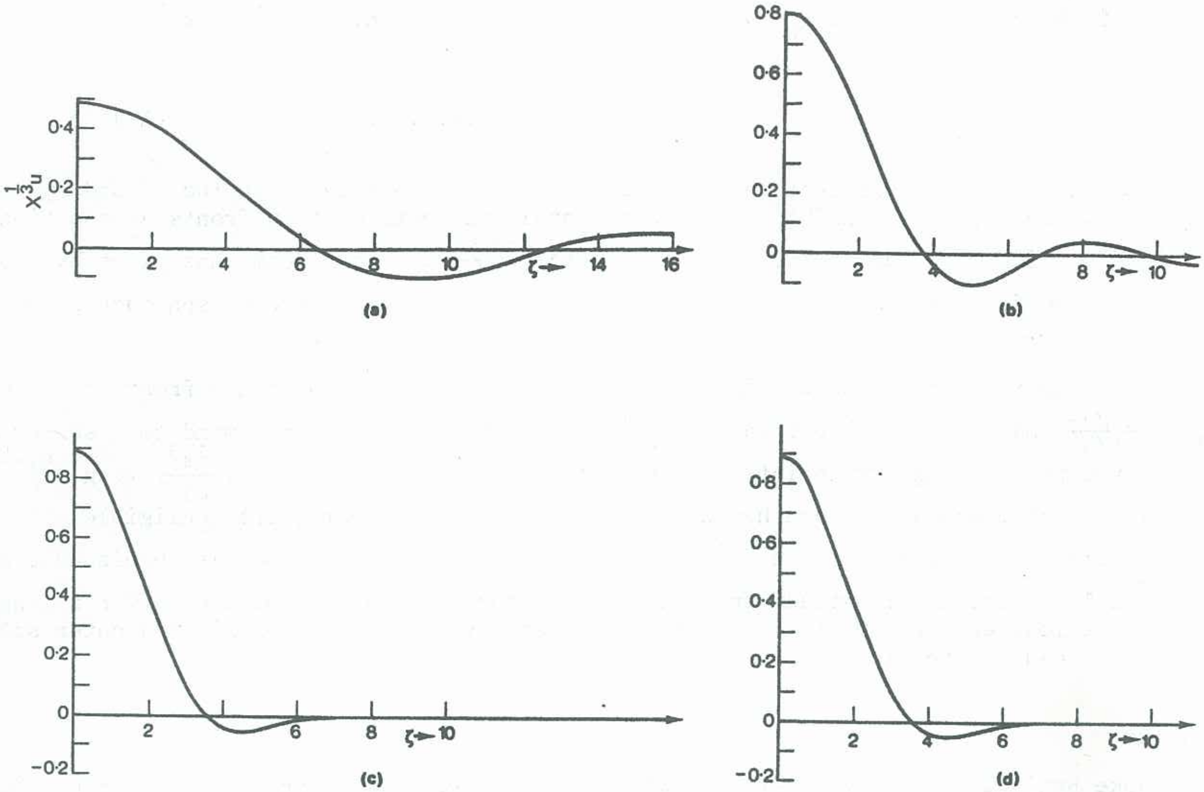

Figure 1: Horizontal velocity profiles given by equation (11) when the volume flux is prescribed. (a) $\tau = 0.5$, (b) $\tau = 1.0$, (c) $\tau = 1.5$, (d) $\tau = 2.0$, where $\tau = t/x^{2/3}$.

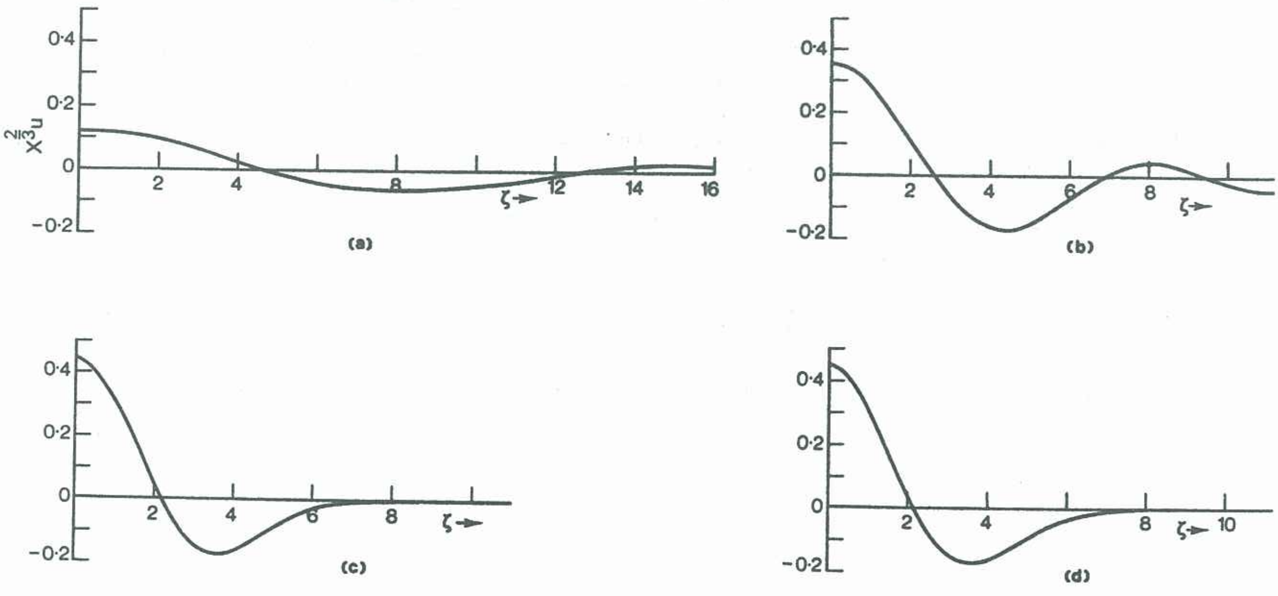

Figure 2: Horizontal velocity profiles given by equation (14) when the flow force is prescribed. (a) $\tau = 0.5$, (b) $\tau = 1.0$, (c) $\tau = 1.5$, (d) $\tau = 2.0$.

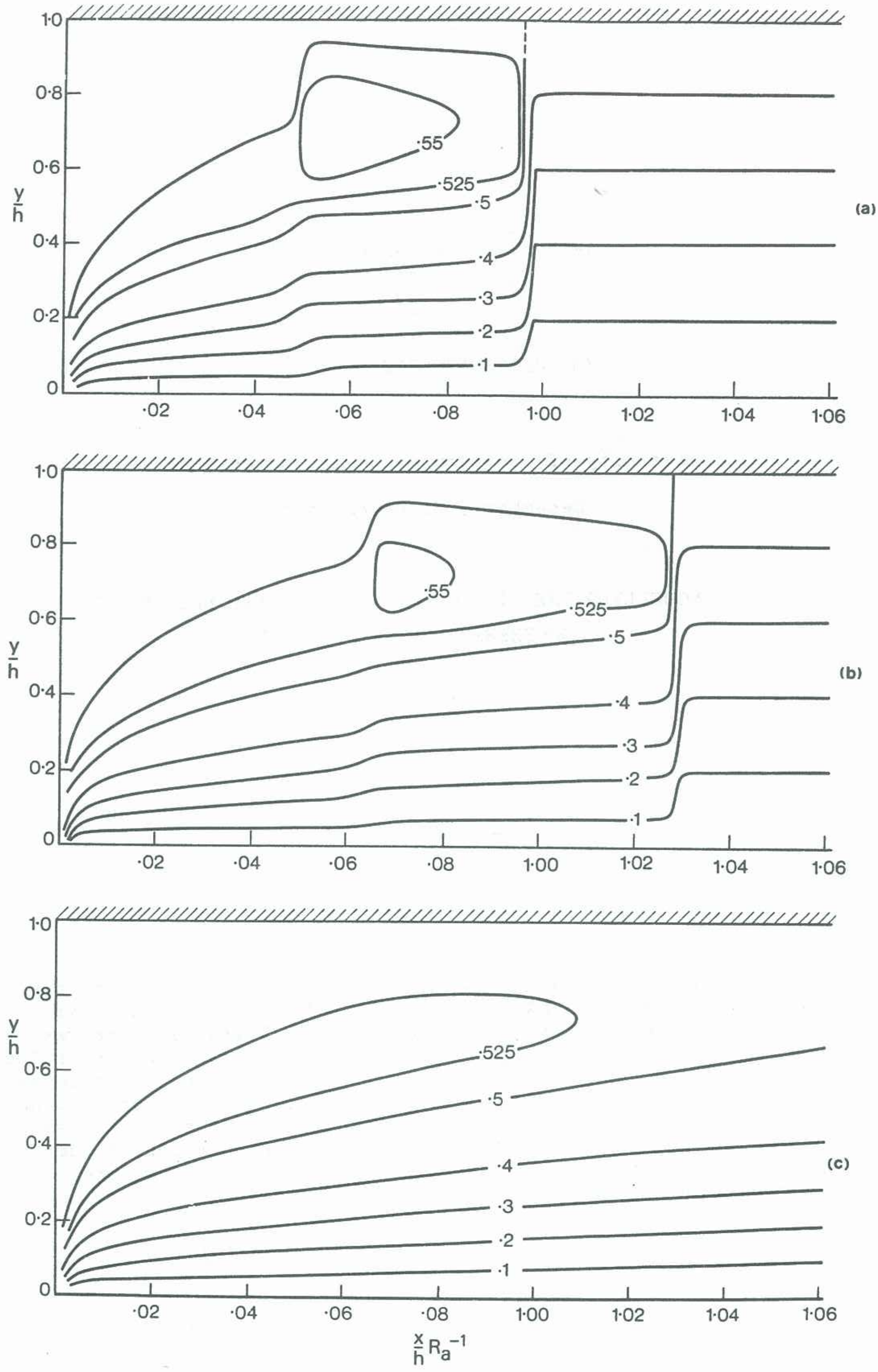

Figure 3: Streamlines in a parallel walled duct given by equation (17). (a) $t^1 = 10^4$ (b) $t^1 = 1.33\times10^4$ (c) $t^1 = 10^6$, where $t^1 = {}^{t}/_{\sqrt{\varepsilon g}}$. The horizontal scale has been compressed by the inverse of the Rayleigh number, R_a^{-1}, where $R_a = \left(\frac{\varepsilon g h^4}{\nu\kappa}\right)^{\frac{1}{2}}$, and has the value 3.16×10^5.