FIFTH AUSTRALASIAN CONFERENCE

on

HYDRAULICS AND FLUID MECHANICS

at

University of Canterbury, Christchurch, New Zealand

1974 December 9 to December 13

LONG WAVE GENERATION ON A NON-UNIFORM SLOPE

by

D.L. CLEMENTS

SUMMARY

The purpose of the present investigation is to obtain analytical solutions to the linearized long-wave equation for cases in which the bottom surface is curved. The technique employed is to transform the relevant governing equation to one associated with either the constant depth or uniform slope situations. All the analytical results for these cases can thus be employed and then the inverse transformations yield information regarding wave propagation in shallow water with a curved bottom profile. This procedure does not, of course, work for an arbitrary bottom profile, but is dependent upon the bottom being specified by certain multi-parameter forms which emerge in the course of the analysis.

D.L. Clements, University of Adelaide

1. INTRODUCTION

The majority of analytical work on two-dimensional wave generation in shallow water has been restricted to either constant depth situations (e.g. Kajiura (1), Momoi (2)) or cases when the bottom has uniform slope (e.g. Tuck and Hwang (3)). In the present paper, we consider the linear shallow water equations for cases when the bottom profile is curved and, in particular, obtain certain profiles for which the equations can be integrated analytically.

2. THE SHALLOW-WATER EQUATIONS WITH GROUND MOTION.

The linear shallow-water equations for arbitrary ground motion are available in Tuck and Hwang (3) and adopt the form

$$\frac{\partial u}{\partial t} + g\,\frac{\partial \eta}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh_0) = \frac{\partial \eta_0}{\partial t}, \tag{2.2}$$

where $u(x,t)$ is the horizontal velocity component. The equations of the bottom and surface are respectively given by

$$y = -h(x,t), \quad y = \eta(x,t) \qquad (2.3),(2.4)$$

in the coordinate system of figure 1 and $\eta_0(x,t)$

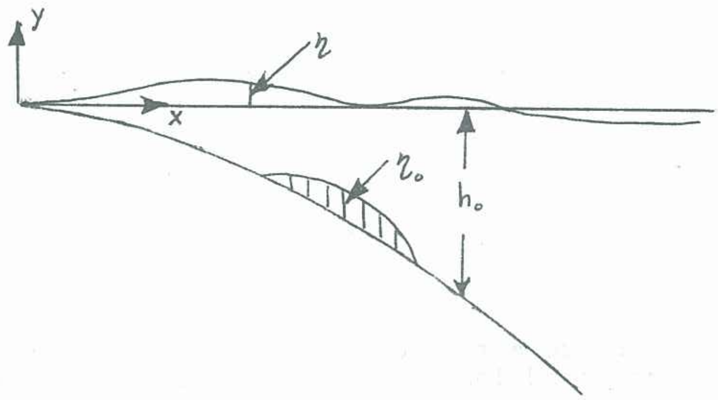

Figure 1

is the magnitude of the upward bottom displacement so that $\partial\eta_0/\partial t$ is the forcing term due to ground motion and

$$h(x,t) = h_0(x) - \eta_0(x,t). \tag{2.5}$$

If we now set

$$\varepsilon = -uh_0(x), \tag{2.6}$$

the system (2.1), (2.2) can be written in the convenient matrix form

$$\Omega_x = \Lambda\Omega_t + \Theta_t \tag{2.7}$$

where subscripts denote partial derivatives and the matrices Ω, Λ and Θ are defined by

$$\Omega = \begin{pmatrix} \eta \\ \varepsilon \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1/(gh_0) \\ 1 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 \\ -\eta_0 \end{pmatrix}. \qquad (2.8)\text{–}(2.10)$$

Introduction of the new independent variables x', t' according to

$$x' = \int [gh_0]^{-\frac{1}{2}}dx, \quad t' = t, \qquad (2.11),(2.12)$$

transforms (2.7) to

$$\Omega_{x'} = H\Omega_{t'} + E_{t'}, \qquad (2.13)$$

where

$$H = \begin{pmatrix} 0 & K^{-\frac{1}{2}} \\ K^{\frac{1}{2}} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ -K^{\frac{1}{2}}\eta_0 \end{pmatrix}, \qquad (2.14),(2.15)$$

with

$$K = gh_0(x),$$

3. THE MATRIX TRANSFORMATIONS

In this section, certain matrix transformations are constructed which transform the system (2.13) to one of two "canonical forms", namely

$$\Omega'_{x'} = H'\Omega'_{t'} + E'_{t'}, \qquad (3.1)$$

where, either

$$\text{(i)} \quad H' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \text{(ii)} \quad H' = \begin{pmatrix} 0 & (g\alpha x')^{-\frac{1}{2}} \\ (g\alpha x') & 0 \end{pmatrix}, \qquad (3.2),(3.3)$$

the case (i) is essentially that corresponding to the constant depth situation, while (ii) is the uniform slope case investigated by Tuck and Hwang [3]. In particular, in the absence of ground motion, for case (i), (3.1) reduces to a form associated with the conventional wave equation. It emerges that such transformations may be constructed subject to $h(x)$ adopting a variety of forms which involve parameters available for curve fitting.

We consider the transformations

$$\Omega'_{x'} = A\Omega_{x'} + B\Omega, \quad |A| \neq 0, \qquad (3.4)$$

$$\Omega'_{t'} = \tilde{A}\Omega_{t'} + \tilde{B}\Omega, \quad |\tilde{A}| \neq 0, \qquad (3.5)$$

where $A, B, \tilde{A}, \tilde{B}$ are 2×2 matrices with entries functions of x' designated by $[a^i_j]$, $[b^i_j]$, $[\tilde{a}^i_j]$ and $[\tilde{b}^i_j]$ respectively. Such transformations are sought which transform

$$\Omega'_{x'} = H'\Omega'_{t'} + E'_{t} \leftrightarrow \Omega_{x'} = H\Omega_{t'} + E_{t'} \qquad (3.6)$$

where H is defined by (2.14) and H' adopts the form (3.2) or (3.3).

The commutativity conditions

$$\Omega'_{x't'} = \Omega'_{t'x'}, \quad \Omega_{x't'} = \Omega_{t'x'}, \qquad (3.7)$$

are satisfied if

$$A = \tilde{A}, \quad \tilde{B}_{x'} = 0, \quad \tilde{B}E_{t'} = 0, \qquad (3.8)\text{-}(3.10)$$

$$\tilde{A}_{x'} - B + \tilde{B}H = 0 \qquad (3.11)$$

while the transformation (3.6) is effected if

$$B = H'\tilde{B}, \quad E'_{t'} = AE_{t'}, \quad H = \tilde{A}^{-1}H'\tilde{A} \qquad (3.12)\text{-}(3.14)$$

It is convenient to choose the matrices $\tilde{A}$ and $\tilde{B}$ in the forms

$$\tilde{A} = \begin{pmatrix} \tilde{a}^1_1 & 0 \\ 0 & \tilde{a}^2_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & \tilde{b}^1_2 \\ \tilde{b}^2_1 & 0 \end{pmatrix}. \qquad (3.15),(3.16)$$

Equation (3.11) now yields

$$(a^1_1)_{x'} - h^1_2{}'\tilde{b}^2_1 + h^2_1{}'\tilde{b}^1_2(a^1_1/a^2_2) = 0, \qquad (3.17)$$

$$(a^2_2)_{x'} - h^2_1{}'\tilde{b}^1_2 + h^1_2{}'\tilde{b}^2_1(a^2_2/a^1_1) = 0. \qquad (3.18)$$

Combination of these equations yields

$$\det A = a^1_1 a^2_2 = \lambda \text{ (constant)}, \quad \lambda \neq 0 \qquad (3.19)$$

so that the system (3.17), (3.18) may be reduced to the single equation

$$(a^1_1)_{x'} + h^2_1{}'\tilde{b}^1_2\lambda^{-1}(a^1_1)^2 - h^1_2{}'\tilde{b}^2_1 = 0. \qquad (3.20)$$

The cases where there is, and is not, ground motion present are discussed in turn in (a) and (b) below.

(a) <u>Ground motion present</u>

In this case equation (3.10) shows that $\tilde{b}^1_2 = 0$ so that (3.20) reduces to

$$(a^1_1)_{x'} - h^1_2{}'\tilde{b}^2_1 = 0. \qquad (3.21)$$

In the case when H' is given by (3.2), $h^1_2{}' = 1$ so that (3.21) yields

$$a^1_1 = \tilde{b}^2_1 x + \delta \qquad (3.22)$$

where δ is a constant. Now from (2.14), (3.14) and (3.22) it follows that

$$h_0 = (g\lambda^2)^{-1}(a^1_1)^4 = (g\lambda^2)^{-1}[\tilde{b}^2_1 x' + \delta]^4 \qquad (3.23)$$

and hence, from (2.11)

$$h_0(x) = (Ax + B)^{\frac{4}{3}}, \qquad (3.24)$$

where A and B are constants with

$$A = (3b^2_1)^{\frac{4}{3}} g^{-1} \lambda^{-\frac{2}{3}}.$$

In the case when H' is given by (3.3), $h^1_2{}' = (g\alpha x')^{-\frac{1}{2}}$ so that (3.21) yields

$$a^1_1 = 2\tilde{b}^2_1(g\alpha)^{-\frac{1}{2}}x'^{\frac{1}{2}} + \delta. \qquad (3.25)$$

Hence, from (2.14), (3.14) and (3.25) we obtain

$$\begin{aligned} h_0 &= (g\lambda^2)^{-1}(g\alpha x')(a^1_1)^4 \\ &= \alpha x'[2\tilde{b}^2_1(g\alpha)^{-\frac{1}{2}}x'^{\frac{1}{2}} + \delta]^4/\lambda^2. \end{aligned} \qquad (3.26)$$

In this case it is not possible, in general, to obtain an explicit form for h_0 in terms of x. However, when $\delta=0$ the simple explicit form

$$h(x) = (Cx+D)^{\frac{6}{5}} \quad (C,D \text{ constants}), \tag{3.27}$$

is obtained.

(b) No ground motion

Here, writing $\alpha = \tilde{b}_2^1\lambda^{-1}$, $\beta = -\tilde{b}_1^2$, reduction to canonical form (i) (equation (3.2)) may be obtained when, by virtue of (3.20)

$$a_1^1 = 1/(\alpha x' + \gamma), \quad \beta = 0, \tag{3.28}$$

$$a_1^1 = -\beta x' + \delta, \quad \alpha = 0, \tag{3.29}$$

$$a_1^1 = (\beta/\alpha)^{\frac{1}{2}}\cot\{(\beta/\alpha)^{\frac{1}{2}}(\alpha x'+\zeta)\}, \ \beta/\alpha > 0, \tag{3.30}$$

$$a_1^1 = (-\beta/\alpha)^{\frac{1}{2}}\tanh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha x'+\eta)\}, \ \beta/\alpha < 0, \tag{3.31}$$

where γ,δ,ζ and η are arbitrary constants. The case (3.29) is, of course, the same as (3.22)

Reduction to canonical form (ii) (equation (3.3)) is only considered in the cases $\tilde{b}_2^1 = 0$ and $\tilde{b}_1^2 = 0$. The former case simply leads to a_1^1 in the form (3.25), while the case $\tilde{b}_1^2 = 0$ gives

$$a_1^1 = \{[2(g\alpha)^{\frac{1}{2}}\tilde{b}_2^1/3\lambda]x'^{\frac{3}{2}} + \mu\}^{-1}. \tag{3.32}$$

From (3.28)-(3.31) together with (2.14) and (3.14) it is seen that reduction to the canonical form (i) may be achieved when h_0 adopts one of the forms

$$\begin{gathered} 1/[\lambda^2 g(\alpha x'+\gamma)^4], \quad [-\beta x' + \delta]^4/(\lambda^2 g), \\ (\beta^2/\alpha^2\lambda^2 g)\cot^4\{(\beta/\alpha)^{\frac{1}{2}}(\alpha x'+\zeta)\}, \ (\beta^2/\alpha^2\lambda^2 g)\tanh^4\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha x'+\eta)\} \end{gathered} \tag{3.33}$$

where the (x',x)-relation is given by (2.11). Similarly, (3.32) corresponds to a reduction to canonical form (ii) when

$$h_0 = \alpha x'\lambda^{-2}\{[2(g\alpha)^{\frac{1}{2}}\tilde{b}_2^1/3\lambda)x'^{\frac{3}{2}} + \mu]^{-4} \tag{3.34}$$

In particular, if $\mu=0$, this yields

$$h_0(x) = (\sigma x + \tau)^{-10/7}, \tag{3.35}$$

where σ,τ are constants.

4. SOLUTIONS TO THE BASIC EQUATIONS

Eliminating u from (2.1) and (2.2) we obtain

$$\frac{\partial^2\eta}{\partial t^2} - g\frac{\partial}{\partial x}\left[h_0\frac{\partial\eta}{\partial x}\right] = \frac{\partial\eta_0}{\partial t}. \tag{4.1}$$

In this section we use of analysis of the preceding sections to obtain solutions to equation (4.1). We consider bottom profiles which are special cases of (3.24) and (3.27). The profiles $h_0 = x^{\frac{4}{3}}$ and $h_0 = x^{\frac{6}{5}}$ are considered separately.

$$\text{(i)} \quad h_0 = x^{\frac{4}{3}} \tag{4.2}$$

In this case (3.24) is applicable with $A=1$ and $B=0$. Using the analysis of section 3 we may show that the transformations

$$\eta' = 3g^{-\frac{1}{2}}x^{\frac{1}{3}}\eta, \quad x' = 3g^{-\frac{1}{2}}x^{\frac{1}{3}}, \tag{4.3),(4.4}$$

transform (4.1) to

$$\frac{\partial^2\eta'}{\partial t^2} - \frac{\partial^2\eta'}{\partial x'^2} = x'\frac{\partial\eta_0}{\partial t} \tag{4.5}$$

Putting $u = (x'+t)/2$, $v = (x'-t)/2$ in (4.5) it follows that

$$\frac{\partial^2\eta}{\partial u\partial v} = -(u+v)\eta_0', \quad (\eta_0' = \partial\eta_0/\partial t) \tag{4.6}$$

Hence

$$\eta' = F(u) + G(v) - \iint (u+v)\eta_0' du dv, \tag{4.7}$$

where F and G are arbitrary functions of their respective arguments. It thus follows that

$$\eta = (1/3)g^{\frac12}x^{-\frac13}\{F(u)+G(v) - \iint (u+v)\eta_0' du dv\}, \tag{4.8}$$

where

$$u = \tfrac12(3g^{-\frac12}x^{\frac13}+t), \quad v = \tfrac12(3g^{-\frac12}x^{\frac13}-t).$$

$$\text{(ii)} \quad h_0(x) = x^{\frac65}$$

In this case (3.27) is applicable with $C=1$, $D=0$ and the results of section 3 show that the transformations

$$\eta' = 2(g\alpha)^{-\frac12}\tilde{b}_1^2 x'^{\frac12}\eta, \quad x' = (5/2)g^{-\frac12}x^{\frac25},$$

transform (4.1) to

$$\frac{\partial^2\eta}{\partial t^2} - \frac{1}{x'^{\frac12}}\frac{\partial}{\partial x'}\left(x'^{\frac12}\frac{\partial\eta'}{\partial x'}\right) = x'^{\frac12}\frac{\partial\eta_0}{\partial t}. \tag{4.9}$$

Introduction of the new independent variable

$$\bar{x} = (2/3)x'^{\frac32} = (2/3)[(5/2)g^{-\frac12}]^{\frac32}x^{\frac35}$$

transforms (4.9) to

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\partial}{\partial\bar{x}}\left[\bar{x}\frac{\partial\eta'}{\partial\bar{x}}\right] = (3\bar{x}/2)^{\frac13}\frac{\partial\eta_0}{\partial t}.$$

The solution to this equation may be expressed in terms of integrals involving Bessel functions (see Tuck and Hwang [3]).

REFERENCES

(1) Kajiura, K. The leading wave in a tsunami. Bull. Earthquake Res. Inst. 41 (1963), 535.

(2) Momoi, T. Tsunami in the vicinity of a wave origin. Bull. Earthquake Res. Inst. 42 (1964), 133.

(3) Tuck, E.O. & Hwang, L.-S. Long wave generation on a sloping beach. J. Fluid Mech. 51 (1972), 499.