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POTENTIAL FLOW INTO AN AXISYMMETRIC DRAIN †

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SUMMARY

Seepage flow into a row of drain pipes is modelled by the following situation. The region $a < r < b$, $|z| < h$ (cylindrical polar coordinates) is filled with an isotropic homogeneous porous medium. Darcy's Law is assumed, so there is an harmonic velocity potential ϕ. On $r = b$ (the surface of the ground) ϕ is zero. On $r = a$, $|z| > \varepsilon h$ (the surface of the drainpipe) and $z = \pm h$ the normal derivative of ϕ is zero and on $r = a$, $|z| < \varepsilon h$ (the gap between the drainpipes) ϕ is a constant, ϕ_0. The aim is to calculate the capacitance of this drain - the flow rate for a given difference in head ϕ_0. The mixed boundary value problem for ϕ can be reduced to a Fredholm integral equation which can be solved numerically. Alternatively it is possible to use the method of matched asymptotic expansions to obtain an asymptotic expression for the capacitance as $\varepsilon \to 0$. To the lowest order, the capacitance is proportional to $-1/\ln \varepsilon$ and is independent of the geometry far away from the gap.

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§1. Introduction.

The aim of this paper is to study seepage flow of water into a row of drainpipes embedded in soil. The soil is assumed to be isotropic and homogeneous so Darcy's law is valid in the form

$$\underset{\sim}{q} = - k\nabla p$$

where $\underset{\sim}{q}$ is the seepage velocity of the water, p the head which drives the flow and k a constant. The flow is assumed steady so $\nabla . \underset{\sim}{q} = 0$ and $\nabla^2 p = 0$.

The drainpipes are hollow circular cylinders of radius a, length 2ℓ, and made of some non-porous material. These pipes are placed at a distance $2d$ apart in a cylindrical tunnel of radius a (see Figure 1). The water enters the drain through the gaps between the pipes and runs away.

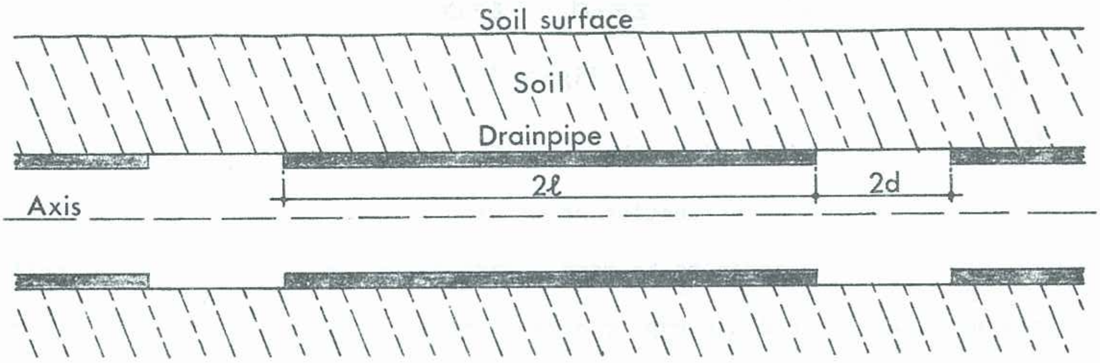

Figure 1

In reality, the surface of the soil is approximately plane, which does present an analytical difficulty - the natural coordinate system for such a problem is cylindrical bipolar, but Laplace's equation does not separate in this system. We avoid this difficulty by assuming that the surface of the soil is represented by a cylinder of radius b coaxial with the surface of the drainpipes.

We seek to calculate the capacitance of the drain - the flow rate per unit difference in head between the soil surface and the inside of the drain. This problem has been considered by Kirkham (1950), who obtained analytically upper and lower bounds for the capacitance; he assumed that the average of the upper and lower bounds approximates the exact value. In §2 we show that the problem may be formulated as a Fredholm integral equation, and describe a simple numerical technique for its solution. In §3 we obtain an asymptotic expression for the capacitance in the limit of narrow gap width.

§2. Formulation of the problem as an integral equation.

We adopt a cylindrical polar coordinate system with axis that of the drain and origin in the middle of one of the gaps. If we neglect end effects ("the drain extends indefinitely in each direction"), we can confine our attention to the finite section $|z| \leqslant \ell + d = h$ say.

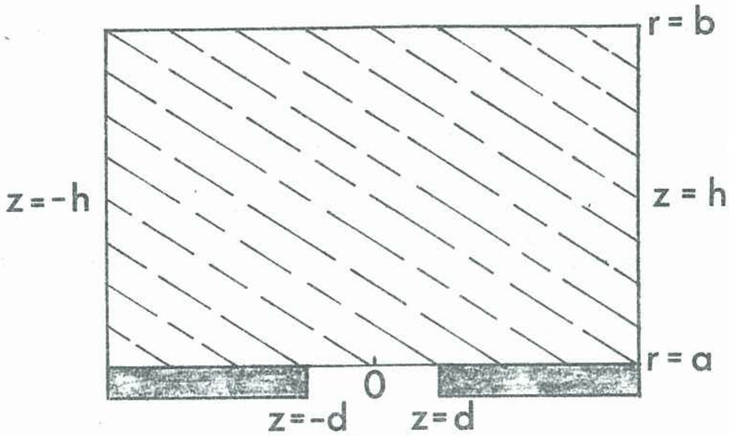

Figure 2

We introduce a velocity potential $\phi = -kp$ for the seepage flow $\underset{\sim}{q}$. Because of the axisymmetry of the problem ϕ is a function of r and z only. On the soil surface the head p and hence ϕ will be taken to be zero. Inside the drain the head is a constant, say $-p_0$, so on the surface of the drain gap $\phi = V_0$ $(= kp_0)$. On the surfaces $z = \pm h$ the normal flow of water is zero; on the surfaces $r = a$, $d < |z| < h$, it is also zero since the drain pipes are non-porous. To summarise $-$ $\phi(r, z)$ satisfies the following conditions:

$$\nabla^2\phi = 0 \qquad 2.1$$

$$\phi(b, z) = 0 \qquad 2.2$$

$$\phi(a, z) = V_0, \; |z| < d \qquad 2.3$$

$$\phi_z(r, \pm h) = 0 \qquad 2.4$$

$$\phi_r(a, z) = 0, \; d < |z| < h \qquad 2.5$$

Since ϕ must be an even function of z, and in view of 2.4, we try a solution of the form

$$\phi = \frac{1}{2}\, a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos\frac{n\pi z}{h} \; ; \qquad 2.6$$

from 2.1 and 2.2,

$$a_0(r) = A_0 \; ln(\frac{b}{r}),$$

$$a_n(r) = A_n \left\{K_0(\frac{n\pi r}{h})I_0(n\pi\lambda_2) - I_0(\frac{n\pi r}{h})K_0(n\pi\lambda_2)\right\} ,$$

where the A_n are constants, K_0 and I_0 are the usual modified Bessel functions and $\lambda_2 = b/h$.

We set

$$\phi_r(a, z) = - f(z),$$

where $f(z) = 0$ for $d < |z| < h$ from 2.5, but is unspecified in the range $|z| < d$. It follows from the theory of Fourier series that

$$A_0 = \lambda_1 \int_{-h}^{h} f(\alpha)d\alpha,$$

and

$$n\pi A_n[K_1(n\pi\lambda_1)I_0(n\pi\lambda_2) + I_1(n\pi\lambda_1)K_0(n\pi\lambda_2)] = \int_{-h}^{h} f(\alpha)\cos\frac{n\pi\alpha}{h}\,d\alpha,$$

where $\lambda_1 = a/h$. Substituting these expressions in 2.6, and interchanging the order of integration and summation, gives

$$\phi(r, z) = \int_{-h}^{h} K(r, z, \alpha)f(\alpha)d\alpha \qquad 2.7$$

where

$$K(r, z, \alpha) = \frac{1}{2}\lambda_1 \, ln(\frac{b}{r}) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{T_n(r)}{n}\cos\frac{n\pi z}{h}\cos\frac{n\pi\alpha}{h},$$

$$T_n(r) = \frac{K_0(n\pi r/h)I_0(n\pi\lambda_2) - K_0(n\pi\lambda_2)I_0(n\pi r/h)}{K_1(n\pi\lambda_1)I_0(n\pi\lambda_2) + K_0(n\pi\lambda_2)I_1(n\pi\lambda_1)}. \qquad 2.8$$

In view of 2.3, 2.5 and 2.7 we have

$$\int_{-d}^{d} K(r, z, \alpha)f(\alpha)d\alpha = V_0, \quad |z| < d, \qquad 2.9$$

a Fredholm integral equation of the first kind, from which $f(\alpha)$ and hence the solution can be determined.

Equation 2.9 can readily be solved numerically. One method is to approximate $f(\alpha)$ by a truncated Fourier cosine series in the interval $(-d, d)$:

$$f(z) = \frac{1}{2}c_0 + \sum_{n=1}^{N} c_n \cos\frac{n\pi z}{d}.$$

On substituting this expression into 2.9 and equating the first $N + 1$ Fourier coefficients of each side, one has $N + 1$ linear equations for determining the constants $c_0, c_1, \ldots, c_N$. A detailed account of this and another numerical method is given in Cheeseman et al. (1973).

The main interest of this problem is the relationship between the capacitance of the drain and the gap width 2d. A dimensionless measure of this capacitance is the "relative flow" Q_R defined by setting $Q_R = Q_d/Q_h$ where

$$Q_d = 2\pi a \int_{-d}^{d} f(\alpha)\,d\alpha$$

is the flow rate of water through the drain, and $Q_h = 4\pi h V_0/\ln(b/a)$ is the flow rate through the corresponding open drain without pipes where $\ell = 0$ and $d = h$. Table 1 includes numerical values of Q_R together with Kirkham's upper and lower bounds.

TABLE 1

$\lambda_1 = 2, \quad \lambda_2 = 4$										
ε	.19	.17	.15	.13	.11	.09	.07	.05	.03	.01
Q_R	.651	.631	.610	.588	.564	.537	.506	.470	.424	.351
Q^*_R	.667	.645	.622	.597	.571	.543	.510	.473	.426	.351
K_L	.603	.586	.567	.547	.526	.502	.475	.442	.401	.335
K_U	.735	.711	.685	.657	.627	.594	.557	.514	.459	.373

$\lambda_1 = 4, \quad \lambda_2 = 10$										
ε	.19	.17	.15	.13	.11	.09	.07	.05	.03	.01
Q_R	.827	.815	.801	.786	.769	.749	.726	.697	.657	.586
Q^*_R	.834	.821	.806	.790	.772	.753	.729	.699	.658	.585
K_L	.796	.784	.771	.757	.741	.723	.701	.673	.635	.568
K_U	.876	.863	.848	.831	.812	.790	.764	.732	.688	.608

Note: Q_R is the numerically calculated value of the relative flow;

Q^*_R is the value of the relative flow calculated from the asymptotic formula 3.6;

K_L is Kirkham's lower bound for the relative flow;

K_U is Kirkham's upper bound for the relative flow.

3. Asymptotic theory.

Since d will be small compared with h in most practical drains, it is useful to find an asymptotic formula for Q_R when d/h is small by the method of matched asymptotic expansions - i.e., by matching an "outer expansion" of ϕ (valid far away from the gap) with an "inner expansion" (valid in the region close to the gap).

Outer expansion

Suppose V_0 is such that the total flow through the drain is $2\pi a$ - i.e., such that

$$\int_{-d}^{d} f(\alpha)d\alpha = 1.$$

We set $\varepsilon = d/h$, introduce dimensionless variables $r' = r/h$, $z' = z/h$, $\alpha' = \alpha/h$, and a dimensionless F by setting $F(\xi) = f(\xi d)d$, so that 2.7 becomes

$$\phi = \int_{-1}^{1} K'(r', z', \varepsilon\xi)F(\xi)d\xi \quad , \qquad 3.1$$

where K' is just K written in terms of the dimensionless variables. If in 3.1 one expands K' as a power series in ε and uses the relationships:

$$\int_{-d}^{d} f(\alpha)d\alpha = \int_{-1}^{1} F(\xi)d\xi = 1, \quad \int_{-1}^{1} \xi F(\xi)d\xi = 0,$$

one finds the outer expansion

$$\phi = K(r', z', 0) + O(\varepsilon^2). \qquad 3.2$$

Inner expansion

For the region close to the gap we introduce the inner dimensionless variables

$$X = (r - a)/d, \quad Z = z/d = z'/\varepsilon.$$

Laplace's equation for ϕ in terms of these inner variables is

$$\frac{\partial^2\phi}{\partial X^2} + \left(\frac{\varepsilon}{\lambda_1 + \varepsilon X}\right)\frac{\partial\phi}{\partial X} + \frac{\partial^2\phi}{\partial Z^2} = 0.$$

If ϕ is expanded as a power series in ε - i.e.,

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots \quad -$$

to zeroth order one has

$$\frac{\partial^2\phi_0}{\partial X^2} + \frac{\partial^2\phi_0}{\partial Z^2} = 0,$$

$$(\phi_0)_{X=0} = V_0(|Z| < 1), \quad (\frac{\partial\phi_0}{\partial X})_{X=0} = 0 \quad (|Z| > 1).$$

In terms of elliptic cylindrical coordinates (u, v) defined by

$$Z = \cosh u \cos v, \quad X = \sinh u \sin v,$$

the zeroth order solution is therefore

$$\phi_0 = V_0 - Au,$$

where A is a constant. The condition that the flow through the gap is $2\pi a$ implies $A = 1/\pi$, so the inner expansion for ϕ is

$$\phi = V_0 - \frac{u}{\pi} + O(\varepsilon). \qquad 3.3$$

Matching the expansions

The constant V_0 can be determined by matching the inner and outer expansions using the "asymptotic matching principle" (Van Dyke 1964). If the inner expansion 3.3 is written in terms of outer variables x' and r' and expanded in powers of ε, one obtains

$$\phi \sim V_0 - \frac{1}{\pi} \ln(\frac{2}{\varepsilon}\sqrt{(r'-\lambda_1)^2 + z'^2}),$$

or in terms of inner variables

$$\phi \sim V_0 - \frac{1}{\pi} \ln(2\sqrt{X^2 + Z^2}) \qquad 3.4$$

The outer expansion 3.2, written in terms of inner variables and expanded in powers of ε, gives

$$\phi \sim \frac{1}{2}\lambda_1 \ln(\lambda) - \frac{2}{\pi^2\lambda_1} \ln 2 - \frac{1}{\pi} \ln(\pi\varepsilon\sqrt{X^2+Y^2}) + \frac{1}{\pi} H(\lambda_1), \qquad 3.5$$

where $\lambda = b/a$ and

$$H(\lambda_1) = \sum_{m=1}^{\infty} \frac{1}{m} [T_m(a) - 1 + \frac{1}{\pi\lambda_1(2m-1)}].$$

According to Van Dyke's asymptotic matching principle the expressions 3.4 and 3.5 must be equal, so that

$$V_0 = \frac{1}{2}\lambda_1 \ln(\lambda) + \frac{1}{\pi} H(\lambda_1) - \frac{2}{\pi^2\lambda_1} \ln 2 + \frac{1}{\pi} \ln(\frac{2}{\pi\varepsilon}). \qquad 3.5$$

The total flow through the corresponding open drain is $4\pi a V_0/(\lambda_1 \ln\lambda)$ whence

$$Q_R = \lambda_1 \, ln\lambda/(2V_0)$$

or,

$$Q_R = \left\{1 + \frac{2}{\pi\lambda_1 \, ln\lambda} \left[ln(\frac{2}{\pi\varepsilon}) + H(\lambda_1)\right] - \frac{4 \, ln \, 2}{\pi^2\lambda_1{}^2 \, ln\lambda}\right\}^{-1} \quad . \qquad 3.6$$

§4. Conclusions

Table 1 shows good agreement between the numerically calculated values of Q_R and those calculated using the asymptotic formula 3.6. Even when ε is as large as .19 the difference is of the order of 2%.

As $\varepsilon \to 0$, to leading order 3.5 becomes

$$V_0 = \frac{1}{\pi} \, ln(\frac{1}{\varepsilon}) \, ,$$

so that the capacitance of the drain is $2\pi a/V_0 = -2\pi^2 a/ln \, \varepsilon$. A similar asymptotic dependence on $ln \, \varepsilon$ in the case of a two-dimensional drain is shown in Equation 9 of Cheeseman et al. (1972). Consequently, to leading order the capacitance is independent of b and h (although of course some outer length scale is implicit in ε). Thus the flow rate through the drain does not depend upon the geometry far away from the gap. It is the region close to the gap that is most important. This accounts for the difference between the upper and lower bounds for Q_R obtained by Kirkham (1949) being larger than expected. (Even with $\varepsilon = 1/1440$ the difference was 3.2%.) Kirkham's upper and lower bounds were obtained by changing the shape of the equipotential surface $\phi = V_0$ - by adding soil to the drain or by removing soil. This changes the geometry of the region around the gap and hence affects the flow rate significantly.

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