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BOUNDARY LAYERS ON A ROTATING DISK

by

Tuncer Cebeci* and Douglas E. Abbott**

SUMMARY

A disk rotating in an infinite quiescent fluid is one of the simplest types of three-dimensional boundary-layer flow. Depending on the Reynolds number based on radius and angular velocity, we may have laminar, transitional, and turbulent boundary layers. According to experiments[1], the flow is laminar if the Reynolds number is less than approximately 1.85×10^5. The flow is transitional for Reynolds number between 1.85×10^5 and 2.85×10^5. It is fully turbulent for Reynolds number greater than 2.85×10^5.

In this paper we discuss the prediction of laminar, transitional and turbulent boundary layers on a rotating disk by an efficient numerical method. The method employs the eddy viscosity concept to model the Reynolds shear stress terms and has been previously used to compute two-dimensional boundary layers[2] and recently three-dimensional boundary layers[3,4]. Results are given for values of the rotational Reynolds number from zero to 2×10^6.

*Adjunct Professor, Mechanical Engineering Dept., California State University, Long Beach, California, U.S.A.
**Professor, Mechanical Engineering Dept., Purdue University, Lafayette, Indiana, U.S.A.

Nomenclature

A	damping-length constant
c_{f_θ}	circumferential local skin-friction coefficient
f	radial stream function
g	circumferential stream function
r	radial coordinate (radius from axis of rotation)
R_r	rotational Reynolds number, $\omega r^2/\nu$
R_θ	Reynolds number based on momentum thickness, $\theta\omega r/\nu$
u,v,w	mean radial, axial, and circumferential velocity components, respectively
y	axial coordinate = distance perpendicular to the disk
γ_{tr}	intermittency factor
ε	eddy viscosity
ε^+	dimensionless eddy viscosity, ε/ν
η	similarity parameter
θ	momentum thickness, $\theta = \int_0^\infty w/\omega r\,(1 - w/\omega r)dy$
μ	dynamic viscosity
ν	kinematic viscosity = μ/ρ
ρ	density
τ	shear stress
ω	angular velocity of disk

Subscripts

i	inner region
o	outer region
w	wall

primes denote differentiation with respect to η

Basic Equations

The boundary-layer equations for steady incompressible three-dimensional, axi-symmetric flow near the rotating disk in the absence of a radial pressure gradient are:

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(ur) + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum in the radial direction

$$u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial y} - \frac{w^2}{r} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} - \rho\overline{u'v'}\right) \tag{2}$$

Momentum in the circumferential direction

$$u\frac{\partial w}{\partial r} + v\frac{\partial w}{\partial y} + \frac{uw}{\partial r} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y} - \rho\overline{v'w'}\right) \tag{3}$$

These equations are subject to the boundary conditions:

$$y = 0 \qquad u = 0 \qquad w = \omega r \tag{4a}$$

$$y \to \infty \qquad u = 0 \qquad w = 0 \tag{4b}$$

Before we solve the above system, we first transform it by using the similarity variable

$$\eta = \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} y \tag{5a}$$

and the dimensionless stream function $f(r,\eta)$ defined by

$$\psi(r,y) = (\nu\omega)^{\frac{1}{2}}\ r^2\ f(r,\eta) \tag{5b}$$

Here

$$ur = \frac{\partial\psi}{\partial y}\ , \qquad vr = -\frac{\partial\psi}{\partial r}$$

With the concept of eddy viscosity, the Reynolds shear stress terms in (2) and (3) can be written as

$$-\rho\overline{u'v'} = \rho\varepsilon\frac{\partial u}{\partial y}\ , \qquad -\rho\overline{v'w'} = \rho\varepsilon\frac{\partial w}{\partial y} \tag{6}$$

With (5) and (6), the system (1)-(3) can be written as

Momentum equation in the radial direction

$$[(1+\varepsilon^+)f'']' + 2ff'' - (f')^2 + (g')^2 = r\left(f'\frac{\partial f'}{\partial r} - f''\frac{\partial f}{\partial r}\right) \tag{7}$$

Momentum equation in the circumferential direction

$$[(1+\varepsilon^+)g'']' + 2fg'' - 2f'g' = r\left(f'\frac{\partial g'}{\partial r} - g''\frac{\partial f}{\partial r}\right) \tag{8}$$

Here primes denote differentiation with respect to η and

$$\varepsilon^+ = \varepsilon/\nu\ , \quad f' = \frac{u}{u_r}\ , \quad g' = \frac{w}{u_r}\ , \quad u_r = \omega r \tag{9}$$

The boundary condition (4) become

$$\eta = 0 \qquad f' = 0 \qquad g' = 1 \tag{10a}$$

$$\eta \to \infty \qquad f' = 0 \qquad g' = 0 \tag{10b}$$

Eddy-Viscosity Formulation

At the present there are several mixing-length and eddy-viscosity formulations being used to model the Reynolds shear-stress terms in the boundary-layer equations. Here we shall use the one developed by Cebeci and Smith[2]. This formulation accounts for various boundary-layer effects such as high and low Reynolds numbers, transitional flows, mass transfer, etc. and has shown to give good results[2]. It has also been extended to three-dimensional boundary layers, again producing good results[3,4]. The extension of this formulation to a rotating disk is presented below.

Due to the composite nature of a turbulent boundary layer, we divide the layer into inner and outer regions and define corresponding eddy viscosity formulas by separate expressions in each region. In the inner region of the boundary layer we define the inner eddy-viscosity by

$$\varepsilon_i = L^2\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial \bar{w}}{\partial y}\right)^2\right]^{\frac{1}{2}} \tag{11}$$

Here

$$L = \kappa y[1 - \exp(-y/A)] \tag{12a}$$

$$\bar{w} = \omega x - w \tag{12b}$$

In (12a) κ is von Karman's constant equal to 0.40 and A is a damping-length parameter given by

$$A = A^+\nu\left(\frac{\tau_w}{\rho}\right)^{-\frac{1}{2}} \tag{12c}$$

with $A^+ = 26$.

In the outer region of the boundary layer we define the outer eddy-viscosity by

$$\varepsilon_o = \alpha\left|\int_0^\infty\left[\omega x - (u^2 + \bar{w}^2)^{\frac{1}{2}}\right]dy\right| \tag{13}$$

Here α is a "universal" constant equal to 0.0168.

To account for the effect of low Reynolds number and to account for the transitional region between a laminar and turbulent boundary, we modify the expressions given by (11) to (13). To account for the low Reynolds number effect, we use the expressions given by Cebeci and Mosinskis[5] and by Cebeci[6]. According to ref. (5), κ and A^+ are functions of Reynolds number given by

$$\kappa = 0.40 + \frac{0.19}{1 + 0.19\,z_2^2} \tag{14a}$$

and

$$A^+ = 26 + \frac{14}{1 + z_2^2} \tag{14b}$$

where $z_2 = R_\theta \times 10^{-3} > 0.3$. According to ref. (6), α is a function of Reynolds number given by

$$\alpha = 0.0168\,\frac{(1.55)}{1 + \Pi} \tag{15a}$$

where

$$\Pi = 0.55\,[1 - \exp(-0.243\,z_1^{\frac{1}{2}} - 0.298\,z_1)] \tag{15b}$$

and $z_1 = (R_\theta/425 - 1)$.

As previously discussed, the flow is fully turbulent when the rotational Reynolds number is greater than 2.85×10^5; for the range of 1.85×10^5 and 2.85×10^5, the flow is transitional. The eddy-viscosity formulation given by (11) through (15) is based on experimental data obtained for fully turbulent flows. For this reason, the formulation should not be used to compute transitional boundary layers unless it is corrected for the intermittent behavior of the boundary layer in the transitional region[7]. In the eddy-viscosity methods the "correction" is usually made by multiplying the inner and outer eddy-viscosity formulas by an intermittency factor. Several authors have tried this approach and have obtained satisfactory agreement with experiment. According to the expression used by Cebeci[8], the intermittency factor for an incompressible turbulent boundary layer with zero pressure gradient is given by

$$\gamma_{tr} = 1 - \exp[-G(x - x_{tr})^2] \tag{16a}$$

where

$$G = 0.835 \times 10^{-3} \left(\frac{u_e^2}{\nu^2}\right) R_{x_{tr}}^{-1.34} , \qquad R_x = \frac{u_e x}{\nu} \tag{16b}$$

By interpreting R_x as a rotational Reynolds number, $R_r \equiv \omega r^2/\nu$, and taking $u_e = \omega r$, we may write (16) as

$$\gamma_{tr} = 1 - \exp[-G(r - r_{tr})^2] , \tag{17a}$$

and

$$G = 0.835 \times 10^{-3} \left(\frac{R_r}{r}\right)^2 R_{r_{tr}}^{-1.34} \tag{17b}$$

In transformed coordinates, it can be shown that the inner and outer eddy viscosity formulas can be written as

$$\varepsilon_i^+ = (\kappa\eta)^2 R_r^{\frac{1}{2}}\left[1 - \exp\left(-R_r^{-\frac{1}{2}} \eta \frac{r}{A}\right)\right]^2 \left[(f'')^2 + (g'')^2\right]^{\frac{1}{2}} \gamma_{tr} \qquad \varepsilon_i^+ \leq \varepsilon_o^+ \tag{18a}$$

$$\varepsilon_o^+ = \alpha R_r^{\frac{1}{2}} \left| \int_0^\infty \left\{1 - [(f')^2 + (1 - g')^2]^{\frac{1}{2}}\right\} d\eta \right| \gamma_{tr} \tag{18b}$$

Here

$$A = \frac{A^+ r}{R_r^{3/4}[(f_w'')^2 + (g_w'')^2]^{\frac{1}{4}}} \tag{19}$$

and κ, A^+, α, and γ_{tr} are given by (14a), (14b), (15), and (17), respectively. Equations (18) are then the eddy viscosity formulation used in the present paper.

Comparison with Experiment

We have used the numerical method of H. B. Keller[9], described in ref. (3) to solve the system given by (7), (8), (10), (18). After writing the basic equations (7) and (8) as a first-order system the derivatives are approximated by centered difference quotients and averages centered at the midpoints of net rectangles or net segments. Arbitrary (nonuniform) meshes are used and second-order accuracy is retained. The nonlinear difference equations are solved by Newton's method using an efficient block-tridiagonal factorization technique. For details see Refs. (2), (3) and (9).

Figures 1 to 3 show the results obtained by our method. Figure 1 shows the mean radial and circumferential velocity profiles for a turbulent flow. The calculations were started as laminar at $r = 0$, and were continued as laminar until $R_r = 1.85 \times 10^5$. At that location the turbulent flow calculations were started by activating the eddy-viscosity formulas in the solution of the governing equations.

Figure 2 shows the comparison of calculated and experimental circumferential skin-friction coefficient, c_{f_θ} defined by

$$c_{f_\theta} = \frac{2\mu}{\rho\omega^2 r^2} \left(\frac{\partial w}{\partial y}\right)_w \tag{20a}$$

which in terms of transformed variables can be written as

$$c_{f_\theta} = \frac{2}{r}\left(\frac{\nu}{\omega}\right)^{\frac{1}{2}} g''_w \tag{20b}$$

The agreement with measured velocity profiles and skin friction is very satisfactory.

Figure 3 shows a comparison of calculated and experimental turbulent mean radial and circumferential velocity profiles. The data is due to Erian and Tong[11]. Shown in the plots are the predictions of ref. (10), which was obtained by a different extension of the Cebeci-Smith eddy-viscosity formulation[2]. Again our results agree well with the experimental data.

In conclusion, the method presented here for predicting the three-dimensional boundary layer that develops on a rotating disk is found to be in good agreement with the available experimental data. The method accounts for complete flows, including laminar, transitional, and turbulent regions. The method is capable of accurate predictions of detailed velocity profiles as well as the conventional boundary-layer parameters.

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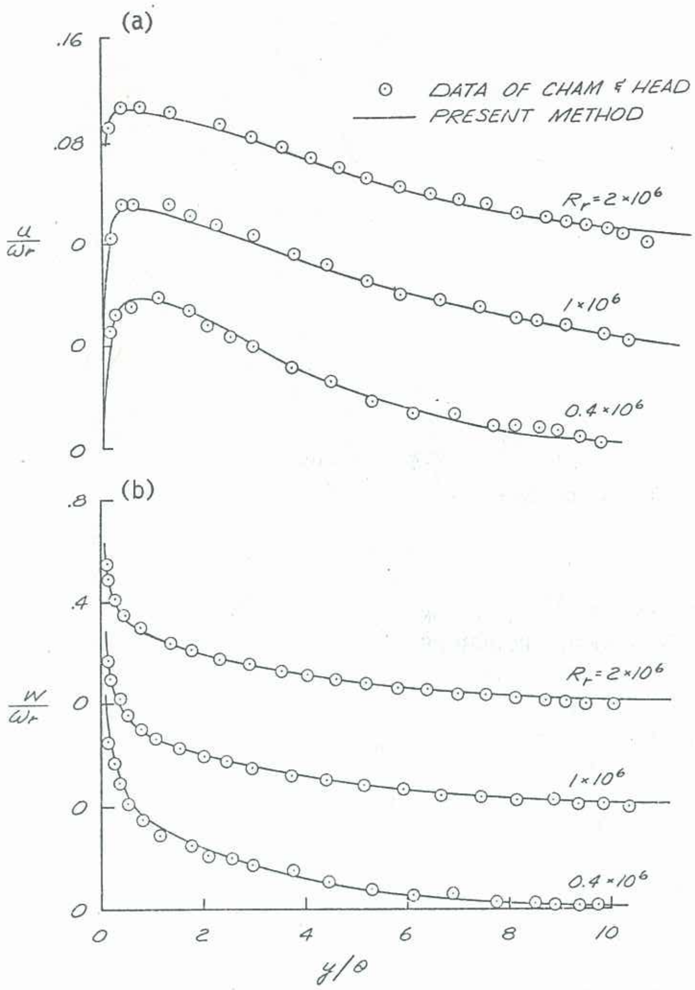

Fig. 1. Comparison of turbulent mean (a) radial and (b) circumferential velocity profiles.

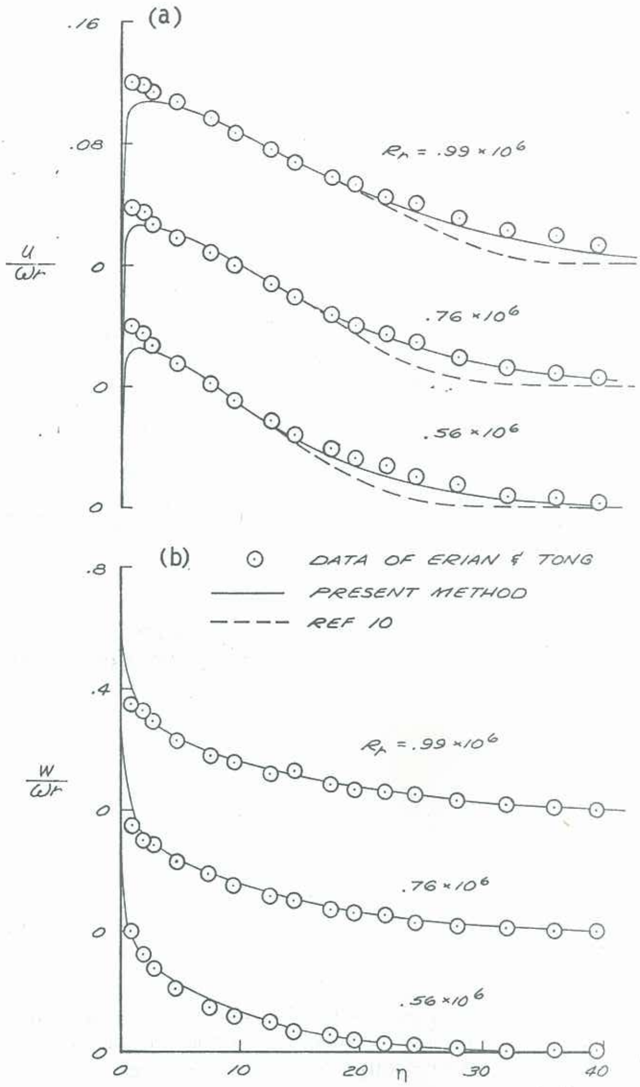

Fig. 3. Comparison of turbulent mean (a) radial and (b) circumferential velocity profiles.

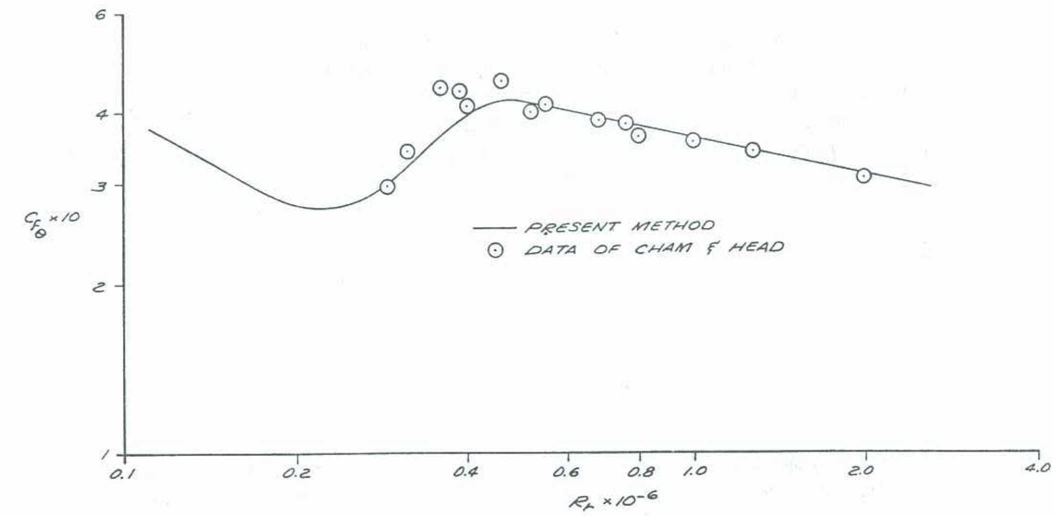

Fig. 2. Comparison of circumferential skin-friction component.