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FLOW OF A LAYERED FLUID FROM A RESERVOIR

by

P.J. Bryant

SUMMARY

This investigation is concerned with the selective withdrawal from a reservoir of a stable layered fluid. Flow takes place through a contraction of rectangular cross-section in which the total volume flow is controlled by a valve at the smallest end. The flow is assumed to be inviscid, steady, and slowly varying in the flow direction. When two layers are being withdrawn, the condition that the flow be continuous at a critical point, where the velocity of waves on the interface is zero, is sufficient to specify the whole flow. The main conclusion drawn from this investigation is that when the valve opening is changed smoothly with the layer depths in the reservoir held constant, there may be discontinuous changes in the flow ratios of the different layers.

Mathematics Department,
University of Canterbury.

1. Introduction

There are many situations in which the fluid in a large reservoir is density stratified, with for example, layers of different temperatures, different salt content, or different sediment load. In such situations, it may be required to withdraw selectively certain layers from the reservoir. The model analysed here is of withdrawal from a reservoir through a contraction of rectangular cross-section into a pipe. This model has previously been investigated theoretically and experimentally by Wood and Lai(1) for the case of one layer flow and for the transition to two layer flow. The present investigation extends their analysis to two layer flow of greater volume flux and to the transition to three layer flow. Certain unsteady flow properties arise here which do not occur in their case.

The present method of analysis is the same in essence as that used by Wood and Lai (1,3) and by Wood (2). Wood (2) examined the discharge of a layered fluid from a reservoir into an open channel, and Wood and Lai (3) studied the discharge of a layered fluid over a broad-crested weir. In all these models, the flow is assumed to be inviscid, steady, and slowly varying in the flow direction. Vertical motion is small compared with horizontal motion, so the pressure is hydrostatic and the flow remains uniform in each layer.

A sketch of the present model, in elevation, is shown in figure 1. The contact point C of the interface BC with the top of the contraction divides the open part from the closed part of the two layer flow. The breadth b of the contraction decreases continuously from large values in the reservoir to its smallest value at the valve V.

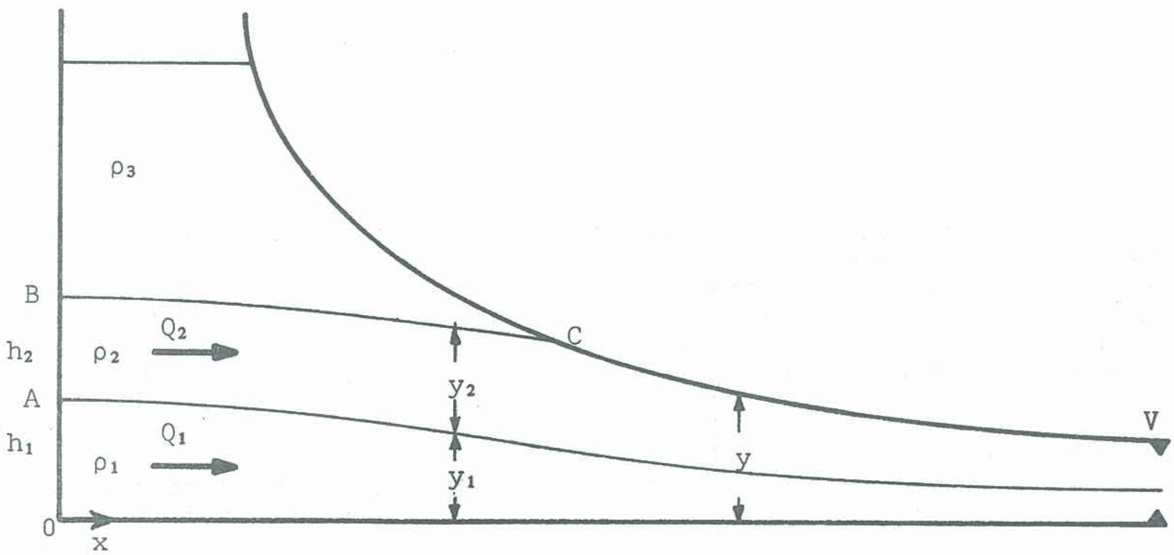

Figure 1. Two layer steady flow through a closed contraction, in elevation. The breadth b of the contraction decreases in the flow direction.

2. Two layer flow

The equations governing two layer flow through the closed part CV of the contraction are an energy equation and a total height equation, namely,

$$\frac{\rho_1 Q_1^2}{2(\rho_1-\rho_2)gb^2y_1^2} - \frac{\rho_2 Q_2^2}{2(\rho_1-\rho_2)gb^2y_2^2} = h_1 - y_1 , \qquad (2.1a)$$

$$y_1 + y_2 = y. \qquad (2.1b)$$

where y,b are known functions of x, and Q_1, Q_2 are volume fluxes. On differentiating these equations with respect to x, then

$$\begin{pmatrix} 1 - \dfrac{\rho_1 Q_1^{\,2}}{(\rho_1-\rho_2)gb^2y_1^{\,3}} & \dfrac{\rho_2 Q_2^{\,2}}{(\rho_1-\rho_2)gb^2y_2^{\,3}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{dy_1}{dx} \\ \dfrac{dy_2}{dx} \end{pmatrix} = \begin{pmatrix} \dfrac{\rho_1 Q_1^{\,2}}{(\rho_1-\rho_2)gb^3y_1^{\,2}} - \dfrac{\rho_2 Q_2^{\,2}}{(\rho_1-\rho_2)gb^3y_2^{\,2}} \\ \dfrac{dy}{db} \end{pmatrix} bx. \qquad (2.2)$$

The equations governing the flow through the open part BC of the contraction are two energy equations, namely,

$$\frac{\rho_1 Q_1^{\,2}}{2(\rho_1-\rho_2)gb^2y_1^{\,2}} - \frac{\rho_2 Q_2^{\,2}}{2(\rho_1-\rho_2)gb^2y_2^{\,2}} = h_1 - y_1, \qquad (2.3a)$$

$$\frac{\rho_2 Q_2^{\,2}}{2(\rho_1-\rho_2)gb^2y_2^{\,2}} = \frac{\rho_2-\rho_3}{\rho_1-\rho_2}(h_1 + h_2 - y_1 - y_2). \qquad (2.3b)$$

Their derivatives satisfy

$$\begin{pmatrix} 1 - \dfrac{\rho_1 Q_1^{\,2}}{(\rho_1-\rho_2)gb^2y_2^{\,3}} & \dfrac{\rho_2 Q_2^{\,2}}{(\rho_1-\rho_2)gb^2y_2^{\,3}} \\ \dfrac{\rho_2-\rho_3}{\rho_1-\rho_2} & \dfrac{\rho_2-\rho_3}{\rho_1-\rho_2} - \dfrac{\rho_2 Q_2^{\,2}}{(\rho_1-\rho_2)gb^2y_2^{\,3}} \end{pmatrix} \begin{pmatrix} \dfrac{dy_1}{dx} \\ \dfrac{dy_2}{dx} \end{pmatrix} = \begin{pmatrix} \dfrac{\rho_1 Q_1^{\,2}}{(\rho_1-\rho_2)gb^3y_1^{\,2}} - \dfrac{\rho_2 Q_2^{\,2}}{(\rho_1-\rho_2)gb^3y_2^{\,2}} \\ \dfrac{\rho_2 Q_2^{\,2}}{(\rho_1-\rho_2)gb^3y_2^{\,2}} \end{pmatrix} bx. \qquad (2.4)$$

The 2 x 2 matrix in equation (2.2) is singular at a critical point, this being the point in the closed part CV where long waves on the interface are stationary. If the flow is smooth at a critical point, that is y_{1x} and y_{2x} both exist there, then the constraints on the flow at the critical point together with the control at the valve V are sufficient to determine the whole flow. For this reason, the critical point is also known as a virtual control. The 2 x 2 matrix of equation (2.4) is singular at a critical point of the open part BC. If there is a critical point in the open part and a critical point in the closed part, the flow cannot be smooth at both points since it would then be over-determined. A single critical point, either in the closed part or in the open part, is sufficient, with the valve control V, to determine the whole flow.

The condition that the flow should be smooth at a critical point in the closed part is that the elements in the 2 rows of the augmented 2 x 3 matrix constructed from equation (2.2) be proportional, that is,

$$1 - \frac{\rho_1 Q_1^{\,2}}{(\rho_1-\rho_2)gb^2y_1^{\,3}} = \frac{\rho_2 Q_2^{\,2}}{(\rho_1-\rho_2)gb^2y_2^{\,3}} = \frac{db}{dy}\left[\frac{\rho_1 Q_1^{\,2}}{(\rho_1-\rho_2)gb^3y_1^{\,2}} - \frac{\rho_2 Q_2^{\,2}}{(\rho_1-\rho_2)gb^3y_2^{\,2}}\right] \qquad (2.5)$$

at the critical point. The 4 equations in (2.1) and (2.5), together with the equation for total volume flux at the valve, provide a solution for the 2 flow constants Q_1, Q_2 and the 3 variables y_1, y_2, x evaluated at the critical point. Once Q_1 and Q_2 are determined, equations (2.1) and (2.3) may be solved for $y_1(x)$, $y_2(x)$ throughout the flow. If, instead, the critical point is in the open part, the elements in the 2 rows of the augmented 2 x 3 matrix constructed from equation (2.4) are proportional, and there are again 5 equations for 5 unknown parameters, from which the solution is determined.

When three layer flow occurs, there are 2 critical points controlling the flow. As above, one set of equations governs the closed part and a second set the open part of the flow. The critical points can occur in either part of the flow alone or one in each part. The constraints imposed on three layer flow by 2 critical points and the valve give rise to 11 equations for 11 unknown parameters. More generally, n layer flow has n - 1 critical points, for which there are $n^2 + n - 1$ equations and unknown parameters.

Consider now an experiment in which the layer depths h_1, h_2, h_3 in the reservoir are held constant while the valve V is given a range of increasing openings. For the smallest openings, the lowest layer flows alone, with the upper layers at rest. When the interface between the first and second layers becomes tangential to the top of the contraction at the contact point, the second layer begins to flow smoothly (1) with the critical point at the former contact point. As the valve is opened further, the critical point moves up the contraction while the new contact point C moves down the contraction. Before the critical point reaches the contact point, the two layer flow breaks down in one of two ways. The first of these is that the interface between the second and third layers becomes tangential to the top of the contraction and the third layer begins to flow smoothly, with its critical points at the position of the two layer critical point and at the position of the two layer contact point C. The second way that the two layer flow can break down is when the open flow at C becomes critical even though the flow is already controlled by a critical point in the closed part. This is the more interesting breakdown and its properties are now discussed.

The two layer flow cannot remain steady and support 2 critical points, since it is then over-determined, so the flow must become unsteady, either temporarily or permanently. An internal jump may be generated at the contact point C and propagate back along the interface towards the reservoir. Alternatively, the two layer flow may change discontinuously into another two layer flow whose only critical point is in the open part, with the closed part supercritical. Alternatively again, the two layer flow may change discontinuously into three layer flow. When the second of these 3 alternatives occurs, and the valve is then opened further, the contact point C moves down the contraction until the interface becomes tangential to the top of the contraction, when three layer flow begins smoothly.

3. Results

The calculations are simplified if the geometry of the contraction is such that bdy/ydb is constant. The calculations here were simplified further by taking b = y, that is, by assuming that the cross-section of the contraction is everywhere square. The non-dimensional volume fluxes, defined by

$$q^2_i = \frac{\rho_i Q_i^{\,2}}{(\rho_1-\rho_2)g(h_1+h_2)^5} , \qquad i = 1, 2, 3$$

are then functions of $h_1/(h_1+h_2)$ and $(\rho_2-\rho_3)/(\rho_1-\rho_2)$. Solutions for q_1, q_2, q_3 against $q_1 + q_2 + q_3$ are sketched in figures 2(a,b) for $(\rho_2-\rho_3)/(\rho_1-\rho_2) = 1$ and $h_1/(h_1+h_2) = 0.2, 0.8$ respectively. The total volume flux $q_1 + q_2 + q_3$ is presumed to be determined by the valve V.

The first curve for q_1 in each figure consists of a section of single layer flow merging continuously into a section of two layer flow having a critical point in the closed part CV of the flow. The second curve for q_1 in each figure consists of a section of two layer flow having a critical point in the open part BC of the flow merging continuously into a section of three layer

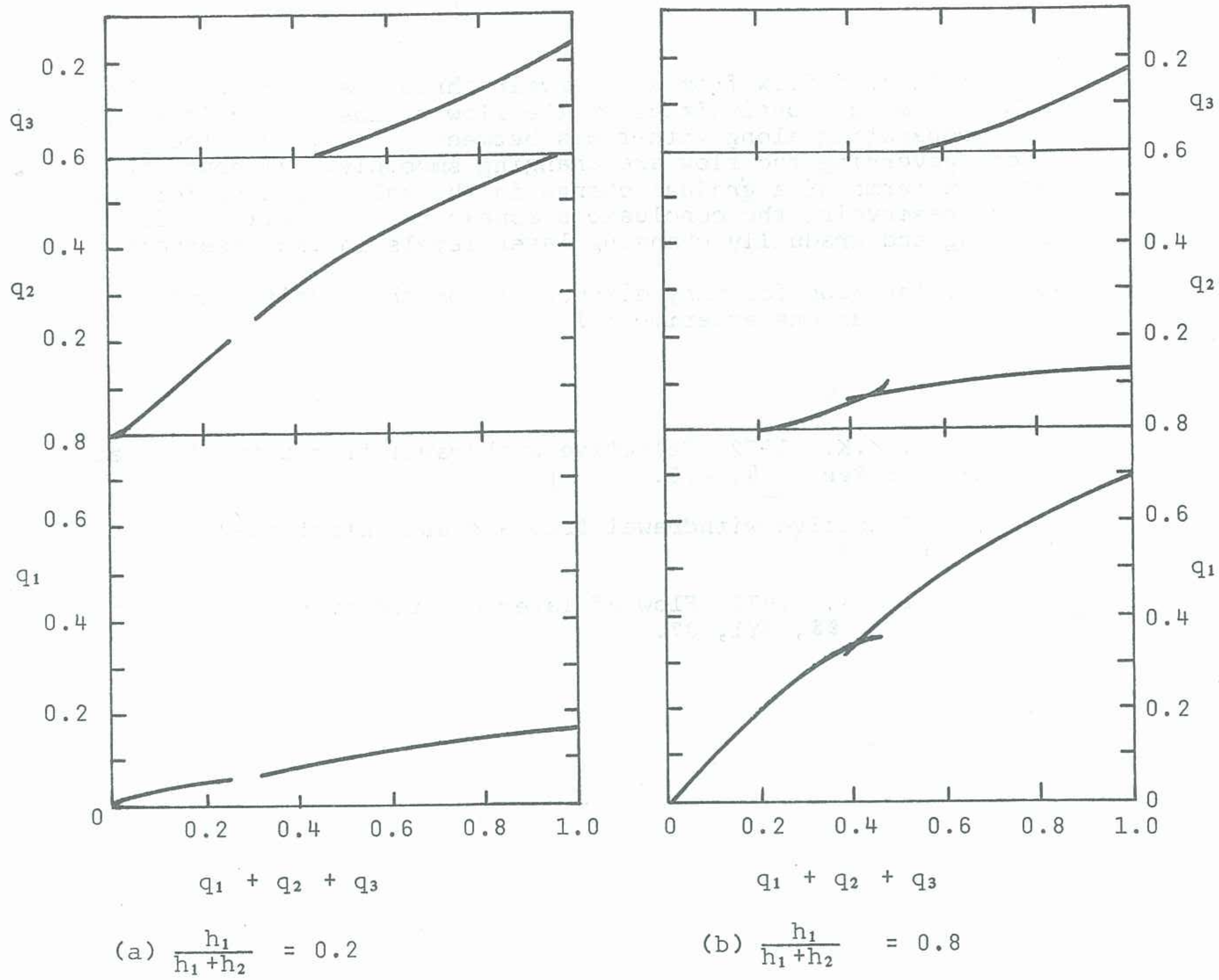

Figure 2. Non-dimensional volume fluxes for one, two, and three layer flow when $(\rho_2-\rho_3)/(\rho_1-\rho_2) = 1$.

flow. The curves for q_2 may be interpreted similarly.

The curves for q_1 and q_2 in figure 2(a) show a gap for $0.26 < q_1+q_2 < 0.32$ in which there is no physically valid steady flow solution. It is suggested that if the valve is given an opening for which the total volume flux lies in this interval, an internal jump will propagate from the contact point C back into the reservoir in order to change the level of the interface to one for which steady flow can occur. This property will be investigated experimentally.

The curves for q_1 and q_2 in figure 2(b) overlap for $0.40 < q_1+q_2 < 0.49$. When the valve is gradually opened, it is suggested that the two layer flow of the first curve persists to the end of the curve near $q_1 + q_2 = 0.49$, since there is no apparent reason for the flow to depart from this curve. There is then a discontinuity in the flow ratio as the flow jumps to the other curve. On the other hand, when the valve is gradually closed starting from three layer flow, it is suggested that the two layer flow of the second curve persists to the end of the curve near $q_1 + q_2 = 0.40$, when there is a discontinuity in the flow ratio. The hysteresis will be investigated experimentally. The cross-over point on the curves at $q_1 = 0.37$, $q_2 = 0.07$ has no apparent significance because the critical points for the two curves lie in different parts of the flow, and although the flow ratio is the same the depth ratio is different.

4. Conclusion

The analysis of layered flow from a reservoir through a contraction has raised the possibility of discontinuities in the flow ratios of the layers, or of internal jumps propagating along interfaces between layers, even though the external conditions governing the flow are changing smoothly. Although the analysis was posed in terms of a gradual change in the valve opening for fixed layer levels in the reservoir, the conclusions appear to be equally valid for a fixed valve opening and gradually changing layer levels in the reservoir.

I am indebted to Ian Wood for many discussions on this subject and for offering to test the conclusions experimentally.

References

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