FIFTH AUSTRALASIAN CONFERENCE

on

HYDRAULICS AND FLUID MECHANICS

at

University of Canterbury, Christchurch, New Zealand

1974 December 9 to December 13

ON THE STABILITY OF CENTRED IMPLICIT DIFFERENCE SCHEMES

by

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SUMMARY

It is found in practice that difference solutions using centred implicit difference schemes are usually unstable if the boundary conditions are not specified as required by the theory of characteristics. This is not in accordance with current stability analysis which claims to prove the unconditional stability of such implicit solutions. This analysis is shown to be fallacious when it fails to investigate the algorithmic structure of the solution, and a modified analysis is shown to give stability criteria in good agreement with the theory of characteristics and experimental evidence.

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Introduction

This paper considers the numerical solution of open channel flows by means of implicit difference schemes. A brief derivation is presented of the simplified homogeneous linear partial differential equations, eliminating the practical complications of non-homogeneous and non-linear instability. These equations are presented in an appropriate form for efficient implicit solutions. It is widely supposed that an analysis given by Richtmyer of the amplification of arbitrary initial errors establishes the unconditional stability of centred implicit difference schemes. This result is criticised on the ground of inconsistency with the theory of characteristics and with empirical experience, and a fallacy in the standard analysis is indicated. A modified analysis is suggested which distinguishes between the alternative algorithmic structures for solving implicit difference schemes. This gives an exact analysis of the single sweep algorithm and an approximate local analysis of the double sweep algorithm.

Differential Equations

For an open channel flow with small slope, negligible flow through the sides and longitudinal flow velocity nearly constant over any cross section at any time, the conservation of momentum and mass can be expressed (e.g. Barnett (2) Chapter 2) as follows:

$$V_t + VV_s + gy_s = g(z_s - S_f)$$

$$A_t + AV_s + VA_s = 0$$

Here A is the cross-sectional area, V the mean velocity, g the acceleration of gravity, and y the "stage" (ordinate of the free surface above a datum axis which is parallel to the channel axis and distance z below some horizontal plane). S_f is the "friction slope" and the subscripts t and s imply partial differentiation with respect to the two independent variables, respectively time and distance along the channel axis.

Now in order to isolate purely numerical effects the flow is taken to be uniform such that $z_s = S_f$ everywhere, and the channel bed is further assumed to be fixed and prismatic so that $A = A(y)$ and $dA/dy = B(y)$, the free surface width. The above equations then take the homogeneous form

$$V_t + VV_s + gy_s = 0$$

$$By_t + AV_s + VBy_s = 0$$

Multiplying the mass equation by $(g/AB)^{\frac{1}{2}}$ and introducing the substitutions $c = c(y) = (gA/B)^{\frac{1}{2}}$, $w = w(y)$ such that $dw/dy = (gB/A)^{\frac{1}{2}}$, produces the simple form

$$V_t + VV_s + cw_s = 0$$

$$w_t + cV_s + Vw_s = 0$$

This form appears more naturally from an examination of the characteristics of the equations but these are of subsidiary interest in this paper.

It is easiest to centre the implicit difference scheme if the equations are rearranged by premultiplying by the inverse of the matrix of coefficients of the space derivatives, giving (cf. Abbott and Ionescu (1)):

$$\begin{bmatrix} V & -c \\ -c & V \end{bmatrix} \begin{bmatrix} V_t \\ w_t \end{bmatrix} + \begin{bmatrix} V^2-c^2 & 0 \\ 0 & V^2-c^2 \end{bmatrix} \begin{bmatrix} V_s \\ w_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Implicit Difference Equations

The variables V and w can now be evaluated at alternate points of a fixed space grid, permitting considerable computational economy. For instance around the j^{th} point the variables appear at the nth time step:

	V	w	V	w
n	×	×	×	×
	j-2	j-1	j	j+1

The difference equations are

$$\frac{V}{2\Delta t}\left(V_j^{n+1} - V_j^n + V_{j-2}^{n+1} - V_{j-2}^n\right) - \frac{c}{\Delta t}\left(w_{j-1}^{n+1} - w_{j-1}^n\right) + \frac{(V^2 - c^2)}{4\Delta s}\left(V_j^{n+1} - V_{j-2}^{n+1} + V_j^n - V_{j-2}^n\right) = 0$$

$$\frac{-c}{\Delta t}\left(V_j^{n+1} - V_j^n\right) + \frac{V}{2\Delta t}\left(w_{j+1}^{n+1} - w_{j+1}^n + w_{j-1}^{n+1} - w_{j-1}^n\right) + \frac{(V^2 - c^2)}{4\Delta s}\left(w_{j+1}^{n+1} - w_{j-1}^{n+1} + w_{j+1}^n - w_{j-1}^n\right) = 0$$

Rearranging

$$\begin{aligned}(F + G)V_j^{n+1} - 2w_{j-1}^{n+1} + (F - G)V_{j-2}^{n+1} &= (F - G)V_j^n - 2w_{j-1}^n + (F + G)V_{j-2}^n \\ (F + G)w_{j+1}^{n+1} - 2V_j^{n+1} + (F - G)w_{j-1}^{n+1} &= (F - G)w_{j+1}^n - 2V_j^n + (F + G)w_{j-1}^n\end{aligned} \qquad (1)$$

where the Froude Number $F = V/c$ has been introduced and

$$G = \tfrac{1}{2}M(F^2 - 1)$$

where the mean Courant Number $M = c\Delta t/\Delta s$.

Conventional Stability Analysis

The difference equations in the form (1) are conventionally investigated for stability by Fourier Series methods, with the unknowns at level n+1 on the left and the known solution at level n on the right. It then follows (cf. Richtmyer and Morton (3) Section 10.3) that this system is unconditionally stable.

Despite the qualified nature of much of Richtmyer's book this result has sometimes been quoted as establishing the practical stability of centred implicit schemes in all circumstances. This is certainly not the case if non-homogeneous terms such as a linear S_f appear in the equations, as may be demonstrated very simply (Barnett (2) Section 4.2), and it is also to be expected that quasi-linear solutions will develop non-linear instabilities in particularly difficult cases which are beyond the reach of linear analysis. However even for the homogeneous linear equations considered here it is found in practice that instability usually develops if the boundary conditions are not specified as required by the theory of characteristics. For instance subcritical flow solutions should require one-point boundary conditions at each end of the channel whereas supercritical flow solutions should require two-point boundary conditions at the upstream end.

Analysis of the Single Sweep Algorithm

The failure of this stability analysis to require the correct specification of boundary conditions raises doubts about its validity. Further, the analysis tacitly assumes that the two difference equations together contain enough information to specify the four unknowns, which is clearly fallacious. In fact if solved by a single sweep from two-point boundary conditions the two upstream unknowns in the system are generated outside the system (1), which may therefore be rewritten as an explicit system:

$$\begin{aligned}(F + G)V_j^{n+1} &= (F - G)V_j^n + (F + G)V_{j-2}^n - (F - G)V_{j-2}^{n+1} - 2w_{j-1}^n + 2w_{j-1}^{n+1} \\ (F + G)w_{j+1}^{n+1} &= -2V_j^n + 2V_j^{n+1} + (F - G)w_{j+1}^n + (F + G)w_{j-1}^n - (F - G)w_{j-1}^{n+1}\end{aligned} \qquad (2)$$

Now, for example, if this system is applied to the upstream boundary with the boundary conditions $w_{j-1}^{n+1} = w_{j-1}^n$ and $V_{j-2}^{n+1} = V_{j-2}^n$ for all n, the first equation reads

$$(F + G)V_j^{n+1} = (F - G)V_j^n + 2GV_{j-2}^n$$

and for the following time step

$$(F + G)V_j^{n+2} = (F - G)V_j^{n+1} + 2GV_{j-2}^n$$

Subtraction gives

$$\frac{V_j^{n+2} - V_j^{n+1}}{V_j^{n+1} - V_j^n} = 1 - \frac{2G}{F + G} = r \quad \text{say}$$

Now taking F as positive there are three possibilities:

(a) G +ve (i.e. $F > 1$) $-1 < r < 1$

(b) G -ve $F > -G$ $r > 1$, $r \to \infty$ as $F \to -G$

(c) G -ve $F < -G$ $r < -1$, $r \to -\infty$ as $F \to -G$

It is at once clear that a necessary condition for the stability of the system (2) is that $F \geq 1$ as otherwise any small variation in the solution of V_j will grow without bound.

The sufficiency of this condition is more difficult to investigate and space does not permit a full treatment. Briefly, it appears that the condition $F > 1$ (i.e. the flow must be supercritical) is also sufficient in the sense that unbounded amplification of errors is inhibited, although in some circumstances intolerably large error amplification may still occur. However it is clear that this analysis reflects the requirements of the theory of characteristics more faithfully than the currently accepted analysis.

Analysis of the Double Sweep Algorithm

In the more usual case of one-point boundary conditions at each end of the channel the system (1) is truly implicit and can be solved only by aggregating J-2 equations for the J unknowns along the channel at time level n+1, and by specifying one unknown at each end as a boundary condition which is not restricted in any way by the difference equations. It follows that a stability analysis of the Double Sweep Algorithm should strictly concern itself simultaneously with all J-2 equations in the remaining J-2 unknowns in a time step. Unfortunately this would greatly reduce the attractiveness of linear stability analysis as such a global analysis would be a great deal more unwieldy than local stability analyses of the Richtmyer type. This is especially true in the many practical cases where the space increments are not all the same. Also while the assumption of local linearity can frequently be justified (Barnett (2) Chapter 5) in practical computations of open channel flows, global linearity would rarely be a realistic assumption. Accordingly the coefficients F and G would have to vary in some manner along the channel at any time, which would preclude the derivation of any simple stability criteria by a global analysis.

The simplicity of local stability analysis can however be retained if the two outer unknowns in the difference system (1) are taken to be generated outside the system by analogy with the above investigation of the Single Sweep Algorithm. This is strictly true if the solution is carried through at only two internal space points, and seems a reasonable first approximation in the more realistic application of many internal space points. Therefore the system (1) is rearranged to

$$(F + G)V_j^{n+1} - 2w_{j-1}^{n+1} = (F - G)V_j^n + (F + G)V_{j-2}^n - (F - G)V_{j-2}^{n+1} - 2w_{j-1}^n$$

$$-2V_j^{n+1} + (F - G)w_{j-1}^{n+1} = -2V_j^n + (F + G)w_{j-1}^n + (F - G)w_{j+1}^n - (F + G)w_{j+1}^{n+1}$$

Solving for V_j^{n+1}, w_{j-1}^{n+1}

$$\begin{bmatrix} V_j^{n+1} \\ w_{j-1}^{n+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{(F - G)T}{F^2 - G^2 - 4} & \frac{2U}{F^2 - G^2 - 4} \\ \frac{2T}{F^2 - G^2 - 4} & 1 + \frac{(F + G)U}{F^2 - G^2 - 4} \end{bmatrix} \begin{bmatrix} V_j^n \\ w_{j-1}^n \end{bmatrix} \qquad (3)$$

where $T = \{F + G - (F - G)a\}X - 2G$

$$U = \{F - G - (F + G)b\}Y + 2G$$

and $a = V^{n+1}_{j-2}/V^{n}_{j-2}$ $\qquad b = w^{n+1}_{j+1}/w^{n}_{j+1}$

$X = V^{n}_{j-2}/V^{n}_{j}$ $\qquad Y = w^{n}_{j+1}/w^{n}_{j-1}$

Unfortunately it is not possible to select such a simple case as that used for the single sweep algorithm. Instead it is necessary to seek necessary stability criteria by following the classical procedure of investigating the eigenvalues of the amplification matrix in (3). The magnitudes of a, b, X and Y are taken as less than or equal to unity to ensure that any amplification of the variables has not been legitimately forced by conditions external to the system. This provides enough of the necessary stability conditions produced by the Fourier series approach of Richtmyer, without necessitating the introduction of Fourier transforms.

The equation for the eigenvalues of the matrix in (3) reduces to

$$(F^2 - G^2 - 4)(1 - \lambda)^2 + \{(F - G)T + (F + G)U\}(1 - \lambda) + TU = 0$$

Clearly, from (3), a necessary condition for stability is

$$F^2 - G^2 - 4 \neq 0$$

which means in practice

$$F^2 - G^2 - 4 < 0 \qquad \text{(I)}$$

Therefore the eigenvalue equation is written

$$(G^2 + 4 - F^2)(\lambda - 1)^2 + \{(F - G)T + (F + G)U\}(\lambda - 1) - TU = 0$$

Therefore

$$\lambda - 1 = \frac{-\{(F - G)T + (F + G)U\} \pm \sqrt{\{(F - G)T - (F + G)U\}^2 + 16TU}}{2(G^2 + 4 - F^2)}$$

A study of all solutions of this equation involves more labour than is justified by the approximate nature of the analysis and by the fact that only necessary stability conditions would be obtained in any case. Fortunately the derivation of necessary conditions from the simple special cases T = 0 and U = 0 reveal many of the characteristics of the general case. Therefore apart from the obvious $\lambda = 1$, two values of λ are considered:

$$\lambda = 1 - \frac{(F - G)T}{G^2 + 4 - F^2} \qquad \text{for } U = 0$$

$$\lambda = 1 - \frac{(F + G)U}{G^2 + 4 - F^2} \qquad \text{for } T = 0$$

For $U = 0$, $\lambda = 1 - \dfrac{(G - F)\{G(2 - (1 + a)X) - F(1 - a)X\}}{G^2 + 4 - F^2}$

For $T = 0$, $\lambda = 1 - \dfrac{(G + F)\{G(2 - (1 + b)Y) + F(1 - b)Y\}}{G^2 + 4 - F^2}$

Considering first $\lambda \simeq 1$, it is reasonable from the definitions of a, b that in this case $a \simeq 1$, $b \simeq 1$ so that taking the two possible values of λ together shows that $\lambda < 1$ implies that $G \pm F$ and G must be of the same sign. In other words

$$F^2 \leqslant G^2 \qquad \text{(II)}$$

is a necessary condition for stability.

Next considering $\lambda \simeq -1$, it is reasonable to take $a \simeq -1$, $b \simeq -1$. For U = 0 the stability condition becomes

$$1 - \frac{(G - F)2(G - FX)}{G^2 + 4 - F^2} \geqslant - 1$$

or

$$F(F - G)(1 + X) \quad \leqslant \quad 4$$

But $U = 0$ implies $G = - FY$ in this case. Thus the condition is

$$F^2(1 + Y)(1 + X) \leqslant 4$$

Since the maximum value of both X and Y is + 1, this means that the condition for stability is

$$F^2 \leqslant 1 \tag{III}$$

For $T = 0$ a similar analysis gives the same result. This condition requires that the double sweep algorithm be used only to solve subcritical flows in either direction, which exactly corresponds with the stability condition indicated by the theory of characteristics. It is also worth noting that either condition (II) or (III) is sufficient to guarantee condition (I).

One more special case is of interest. This is the case where X and Y are both small compared with unity such that all products and squares of X and Y can be neglected. This corresponds with amplitudes an order of magnitude larger at the internal mesh points than at the assumed local boundary points. In this case the expression under the square root in the eigenvalue equation reduces to

$$8G^2\{FG(X(1 + a) - Y(1 + b)) - 2F^2(aX + bY) + 4(X(1 + a) + Y(1 + b))\} - 16G^2(4 - F^2) \\ + 8FG(4 - F^2)(X(1 - a) - Y(1 - b))$$

Assuming conditions (II) and (III) hold this expression is dominated by the term $16G^2(F^2 - 4)$ when X and Y are small and is therefore negative in sign. Accordingly the square root is imaginary. If the eigenvalue equation is written

$$\alpha(\lambda - 1)^2 + \beta(\lambda - 1) + \gamma = 0$$

$$\lambda - 1 = \frac{- \beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

As in this case $\beta^2 - 4\alpha\gamma$ is negative

$$\lambda = 1 - \frac{\beta}{2\alpha} \pm \frac{i\sqrt{4\alpha\gamma - \beta^2}}{2\alpha}$$

$$|\lambda|^2 = 1 - \frac{\beta - \gamma}{\alpha}$$

Substituting,

$$|\lambda|^2 = 1 - \frac{F^2\{X(1 - a) + Y(1 - b)\} + 2FG\{X - Y\} + G^2\{X(1 + a) + Y(1 + b)\}}{G^2 + 4 - F^2}$$

Therefore if X and Y are both negative there is always some small amplification but if the errors at the local boundary points are assumed to be restricted to an absolute maximum, for instance a maximum round-off error, then as the errors are amplified at the internal space points their ratio to the local boundary errors will tend to infinity, or in other words X and Y will tend to zero. At this point the amplification eigenvalue magnitude tends to unity so that further amplification is inhibited. In this restricted case therefore, that is, when errors at the two inner points of the difference system are an order of magnitude larger than the errors at the two outer points, the stability of a further solution is conditional only on the condition (I). This situation bears some resemblance to Richtmyer's result of unconditional stability, but is not particularly helpful as the only circumstances under which this extended stability range can be guaranteed is the rather trivial case where the two outer points are actual boundary points of a four space-point implicit solution.

Numerical Experiments

A simple computer program was set up to solve the given linear homogeneous equations by the double sweep algorithm (Abbott and Ionescu (1)). Various small initial and boundary disturbances were tried with four, ten and in one case twenty space points. As predicted above, solutions with four space points showed initial amplification of errors in some cases, particularly supercritical flows, but all solutions satisfying condition (I) quickly settled to a pattern of negligible error amplification.

All supercritical flow solutions with ten or twenty space points exhibited rapid error amplification which gave no sign of being bounded, at any rate before the solution produced meaningless negative depths. This was in full agreement with the theory of characteristics and with stability condition (III).

All experiments with subcritical flow solutions settled to a pattern of negligible error amplification, including those which did not satisfy stability condition (II). The infringement of condition (II) led to initial amplification of errors but this quickly stabilised. Further consideration of the analysis leading to condition (II) indicates that the amplification eigenvalue concerned is never greatly in excess of unity, and it appears in practice that the slow local growth of errors in a subcritical flow solution tends to produce the situation

$$a \simeq b \simeq X \simeq Y \simeq 1$$

Which means that T and U are both small and the amplification eigenvalue is close to unity. In this case the amplification factor could well vary irregularly between just below unity and just above unity, depending on slight variations of a, b, X and Y.

Conclusions

Conventional stability analysis, which claims to prove the unconditional stability of centred implicit difference solutions of hyperbolic problems, is fallacious in that it fails to investigate the algorithmic structure of the solution. If the difference scheme is solved in a single sweep from a two-point boundary condition it is easily demonstrated that reasonable initial and boundary data give unbounded instability in a subcritical flow solution, so that a necessary condition for stability is that the solution flow be supercritical.

The double sweep algorithm from two one-point boundary conditions is difficult to analyse exactly because it is strictly necessary to consider the whole simultaneous solution for one time step. However an approximate local analysis suggests that a necessary condition for stability is that the solution flow be subcritical, which is in conformity with both the theory of characteristics and numerical experiments. Another necessary condition (condition (II)), which effectively bounds the Courant Number further away from zero as the Froude Number increases, is found in practice to cause little difficulty as its violation usually results in very limited error amplification.

The requirement that boundary conditions be properly specified in accordance with the theory of characteristics therefore dictates the choice of solution algorithm for centred implicit schemes. Such schemes are also by no means free of amplitude error as is commonly supposed.

References

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Acknowledgements

This topic was investigated under a New Zealand National Research Advisory Council Fellowship at the International Courses in Hydraulic Engineering, Delft, The Netherlands by the kind permission of the Director, Professor L. J. Mostertman. The encouragement and advice of Dr M. B. Abbott is gratefully acknowledged.