FIFTH AUSTRALASIAN CONFERENCE

on

HYDRAULICS AND FLUID MECHANICS

at

University of Canterbury, Christchurch, New Zealand

1974 December 9 to December 13

CHARACTERISTICS OF A TWO-DIMENSIONAL ORIFICE-JET PAST A RECTANGULAR PLATE

by

Anat Arbhabhirama and Tang-Huei Wang

SUMMARY

Characteristics of a free jet issuing from a two-dimensional orifice past a rectangular plate placed symmetrically and normal to the flow were studied. The analysis was made using Zhukovskii's method of free streamlines, the application of the Schwarz-Christoffel theorem, and a technique of integration across free streamlines. Theoretical equations were obtained to express the coefficient of contraction and the angle of deflection as functions of the boundary geometry. The solutions of some limiting cases of the problem are verified by experimental results from a previous investigator.

Anat Arbhabhirama, Associate Professor, Asian Institute of Technology, Bangkok, Thailand
Tang-Huei Wang, Inspector, National Economic Council, Taipei, Taiwan.

NOTATION

The following symbols are used in this paper:

A,B,C,D,E,F,G B',C',D',E'	Points on the boundary of the flow pattern
a	Width of the slot
b	Half width of the channel
c	Half width of the rectangular plate
d	Width of the free jet
f,g,k	Coordinates of points in the t-plane
i	Imaginary unit, $i^2 = -1$
ℓ	Distance from one corner of the channel to the sharp edge of the slot
M,N	Complex constants
q	Absolute velocity of flow
t	Complex variable
V_j	Velocity of the free jet
V_m	Maximum velocity occurring along the boundary of the channel
V_o	Uniform velocity in the channel
w	Complex potential
X	The displacement of the rectangular plate from the slot, or the horizontal eccentricity of the channel
x	Horizontal coordinate
y	Vertical coordinate
z	Physical plane, $z = x + iy$
C_c	Coefficient of contraction
β	Angle of deflection of the jet
θ	Angle of velocity vector with respect to x-axis
μ	Dimensionless term, $\mu = V_o/V_j = C_c a/b$
Ω	Complex variable
K(m)	Complete elliptic integral of the first kind with modulus m
$\pi_o(m,n)$	Complete elliptic integral of the third kind with modulus m and parameter n

INTRODUCTION

The pattern of a free jet issuing from an opening can be analyzed using Helmholtz-Kirchhoff theory of free streamlines. The free jet can be characterized by its angle of deflection β, and coefficient of contraction C_c. These characteristics can be determined from the geometry of the efflux section without knowing the precise formation of the whole flow pattern.

In this study, the characteristics of a free jet issuing from a two-dimensional orifice past a rectangular plate placed near the end of the orifice as shown in Fig.1a will be considered. Due to symmetry the problem is analogous to the flow issuing from a two-dimensional slot placed at the end of an approach channel as shown in Fig.1b.

In Fig.1b, a uniform flow of velocity V_o in an approach channel is flowing through a two-dimensional slot placed at the end of the approach channel. The jet issuing from the slot becomes uniform with velocity V_j, thickness d, and the angle of deflection β with respect to x-axis.

Some limiting cases of this problem were studied by von Mises (5), Chowchuvech (3) and Arbhabhirama (1,2).

von Mises (5) studied the case of flow from a two-dimensional slot placed symmetrically at the end of an approach channel, which is a special case of the problem shown in Fig.1b, for $X = 0$ and $c = \ell$. He obtained the relationship between the coefficient of contraction and the boundary configuration of the channel. In his study, a hodograph plane was used to map the original boundaries in the physical plane onto a semicircle of unit radius in the intermediate plane. A differential equation relating the complex potential and the complex variable in the intermediate plane was obtained by balancing sources and sinks. The final solution of the problem was acquired by integrating the differential equation across the opening.

Gurevich (4) reported that the coefficient of contraction calculated by von Mises agreed well with that of Weisbach's experiments.

Chowchuvech (3) experimentally investigated the case where the slot is placed unsymmetrically at the end of the approach channel (i.e. $X = 0$). By measuring the slot opening a, eccentricity $e = \frac{b-a}{2} - c$, jet trajectory, velocity of the free jet V_j, and the discharge of the flow, she was able to determine the coefficient of contraction and jet deflection angle as functions of boundary geometry.

Arbhabhirama (1) theoretically investigated the problem considered by Chowchuvech (i.e. $X = 0$). In his analysis however, he failed to take into account the point where the maximum velocity occurred along the boundary of the approach channel. This restricted his analysis to the case where the maximum velocity occurred at the uniform flow zone in the approach channel.

In this study, Zhukovskii's method of free streamlines, the application of the Schwarz-Christoffel theorem, and a technique of integration across free streamlines suggested by Arbhabhirama (2) will be used to obtain the analytical solution of the problem. The analysis involves a series of transformations from the physical plane to the complex potential plane. For the case when $X = 0$ the theoretical solution will be verified by experimental results obtained by Chowchuvech (3).

Zhukovskii's Method

This method is concerned with the case in which the region occupied the fluid consists of a finite number of straight lines. In lieu of using the hodograph plane in the Helmholtz-Kirchoff method, Zhukovskii as cited by Gurevich (4) introduced

$$\Omega = \ell n\,(V_j dz/dw) = -\ell n\,\frac{q}{V_j} + i\theta \qquad \text{---------------------(1)}$$

where V_j is the velocity of the free jet, q is the magnitude of velocity and θ is the angle of the velocity vector to the x-axis. Eq.(1) transforms the original boundaries to a polygon in the Ω-plane. The polygon can then be opened onto the upper half of the t-plane, and the function $\Omega(t)$ is obtained from the Schwarz-Christoffel theorem. The flow pattern in the t-plane is equivalent to flow from a source to a sink placed on the real axis of the t-plane, thus allowing the determination of $w(t)$. From knowledge of $\Omega(t)$ and $w(t)$ a differential equation relating z and t is found from Eq.(1)

$$dz = \frac{e^{\Omega}}{V_j}\frac{dw}{dt}\,dt \qquad \text{-----------------------(2)}$$

Two parametric equations describing the bulk characteristics can be derived by equating real and imaginary parts of the integration of Eq.(2).

Technique of Integration across Free Streamlines

Arbhabhirama (2) developed this technique to modify Zhukovskii's method in obtaining the parametric equations, which is summarized below.

Consider a jet discharging from a two-dimensional slot as shown in Fig.2a. Let points C and E map onto the t-plane at t_1 and t_3, respectively. Since D and D' are in the uniform flow region, this maps onto a point sink on the real axis as shown in Fig.2b.

Along the free streamlines, $q = V_j$ and $\Omega = i\theta$, Eq.(2) becomes

$$dz = dx + idy = \frac{e^{i\theta}}{V_j}\frac{dw}{dt}\,dt = \frac{\cos\theta + i\sin\theta}{V_j}\frac{dw}{dt}\,dt \qquad \text{------------------(3)}$$

Equating the real and imaginary parts of Eq.(3) yields

$$dx = \frac{\cos\theta}{V_j}\frac{dw}{dt}\,dt \qquad (4)$$

$$dy = \frac{\sin\theta}{V_j}\frac{dw}{dt}\,dt \qquad (5)$$

Integration across the slot from C to E can be separated into three steps viz; from C to D, from D to D', and from D' to E. From Fig.2a, the integrations of dx and dy from D to D' can be geometrically determined. Thus

$$\int_D^{D'} dx = -d\sin\beta; \qquad \int_D^{D'} dy = d\cos\beta \qquad (6)$$

or

$$\int_D^{D'} dz = -d(\sin\beta - i\cos\beta) \qquad (7)$$

Hence, integrating Eqs.(4) and (5) across the slot yields

$$\int_C^{E'} dx - \int_D^{D'} dx = X + d\sin\beta = \mathrm{Pr.v}\int_{t_1}^{t_3} \frac{\cos\theta}{V_j}\frac{dw}{dt}\,dt \qquad (8)$$

and

$$\int_C^{E} dy - \int_D^{D'} dy = a - d\cos\beta = \mathrm{Pr.v}\int_{t_1}^{t_3} \frac{\sin\theta}{V_j}\frac{dw}{dt}\,dt \qquad (9)$$

where the abbreviation Pr.v represents "Cauchy Principle value of".

SOLUTION OF PROBLEM

Four successive conformal mapping planes are used to transform the physical z-plane to the complex potential w-plane. The relation between corresponding points in these various planes is shown in Fig.1.

First Transformation

The transformation from the z-plane to the Ω-plane is represented by Eq.(1). The straight boundaries in the z-plane are streamlines with constant θ, thus are transformed to horizontal lines in the Ω-plane. The boundaries in the z-plane are then transformed into a polygon in the Ω-plane as shown in Fig.1c.

Point G is the point where the maximum velocity V_m occurs along the boundaries AB and AF in the approach channel, $q = V_m$, $\theta = 0$, and $\Omega = -\ell n(V_m/V_j)$. This point is mapped onto the real axis, and therefore becomes a vortex of the polygon in the Ω-plane. It should be noted that leaving the point G out of the Ω-plane as done by Arbhabhirama (1) is equivalent to assuming that the maximum velocity along the boundaries always occurs at the uniform velocity zone, $q = V_o$. This assumption is valid only in the cases of a symmetrical slot, $X = 0$, $c = \ell$, and a slot located on one side of the approach channel, $X = 0$, $c = 0$.

Second Transformation Ω-plane to t-plane

The polygon in the Ω-plane is opened onto the upper half of the t-plane by applying the Schwarz-Christoffel theorem,

$$\Omega = M\int\frac{dt}{(t^2-1)\sqrt{(t^2-k^2)}} + N ,$$

and integration yields

$$\Omega = M\tan^{-1}\frac{t\sqrt{k^2-1}}{\sqrt{t^2-k^2}} + N \qquad (10)$$

Using the boundary conditions at point E, $\Omega = i\pi/2$, $t = -k$, and at point C, $\Omega = -i\pi$, $t = k$, M and N can be determined as $M = -i$ and $N = 0$. Substituting M and N into Eq.(10) gives

$$\Omega = -i\tan^{-1}\frac{t\sqrt{k^2-1}}{\sqrt{t^2-k^2}} \qquad (11)$$

The value of g can be obtained using the condition at point D, $\Omega = i\beta$, $t = -g$. Substituting this condition in Eq.(11) and after simplification gives

$$g = \frac{k \sin \beta}{\sqrt{1-k^2\cos^2\beta}} \qquad \text{------------------------(12)}$$

The relationships among k, f and $\mu = V_o/V_j$ can be determined using Eq.(11) and the condition at point A. Eq.(11) can be rewritten as

$$\Omega = \frac{1}{2}\,\ell n\,\frac{\sqrt{t^2-k^2}+t\sqrt{1-k^2}}{\sqrt{t^2-k^2}+t\sqrt{1-k^2}} \qquad \text{------------------------(13)}$$

and noting that at point A, $\Omega = -\ell n\ V_o/V_j = -\ell n\ \mu$, $t = -f$, Eq.(13) becomes

$$-\ell n\ \mu = \frac{1}{2}\,\ell n\,\frac{\sqrt{f^2-k^2}+f\sqrt{1-k^2}}{\sqrt{f^2-k^2}-f\sqrt{1-k^2}} \qquad \text{------------------------(14)}$$

Simplifying Eq.(14), gives

$$\mu + \frac{1}{\mu} = \frac{\sqrt{f^2-k^2}}{k\sqrt{f^2-1}} \qquad \text{------------------------(15)}$$

<u>Third Transformation</u> w-plane to t-plane

The flow in the t-plane is equivalent to flow to a sink of the half strength $V_o b/\pi$ at D from a source of equal strength at A, thus

$$w = \pm \frac{V_o b}{\pi}\,\ell n\,\frac{t+f}{t+g} \qquad \text{---------------------------(16)}$$

<u>Derivation of Equations</u>

Along a free streamline, $\Omega = i\theta$, substituting in Eq.(11)

$$i\theta = -i\tan^{-1}\frac{t\sqrt{1-k^2}}{\sqrt{k^2-t^2}}$$

thus

$$\sin\theta = \frac{-t\sqrt{1-k^2}}{k\sqrt{1-t^2}} \quad ; \quad \cos\theta = \frac{\sqrt{k^2-t^2}}{k\sqrt{1-t^2}} \qquad \text{-----------------------(17)}$$

Integration across streamlines can be made by substituting the expressions of cosθ and sinθ from Eq.(17) and dw/dt of Eq.(16) into Eqs.(8) and (9).

Thus

$$-X + d\sin\beta = \frac{\mu b}{\pi k}\int_{-k}^{k}\frac{f-g}{(t+f)(t+g)}\,\frac{\sqrt{k^2-t^2}}{\sqrt{1-t^2}}\,dt\ , \qquad \text{-----------------(18)}$$

and

$$a - d\cos\beta = -\frac{\mu b}{\pi k}\int_{-k}^{k}\frac{f-g}{(t+f)(t+g)}\,\frac{t\sqrt{1-k^2}}{\sqrt{1-t^2}}\,dt \qquad \text{-----------------(19)}$$

Eq.(18) can be evaluated as follows:

$$-X + d\sin\beta = \frac{2\mu b}{\pi k}\Bigg[(f-g)\int_{o}^{k}\frac{dt}{\sqrt{(1-t^2)(k^2-t^2)}} + f(k^2-f^2)\int_{o}^{k}\frac{dt}{(f^2-t^2)\sqrt{(1-t^2)(k^2-t^2)}}$$

$$-g(k^2-g^2)\int_{o}^{k}\frac{dt}{(g^2-t^2)\sqrt{(1-t^2)(k^2-t^2)}}\Bigg]$$

$$= \frac{2\mu b}{\pi k}\left[(f-g)K(k) + \frac{k^2-f^2}{f}\,\pi_o\!\left(k, \frac{k^2}{f^2}\right) - \frac{k^2-g^2}{g}\,\pi_o\!\left(k, -\frac{k^2}{g^2}\right)\right] \qquad \text{-------------(20)}$$

In Eq.(20), K(m) is the complete elliptic integral of the first kind with modulus m, and $\pi_o(m,n)$ is the complete elliptic integral of the third kind with modulus m and parameter n. Eq.(20) after simplification can then be written as

$$\frac{X}{b} = \mu\sin\beta + \frac{2\mu}{\pi k}\left[(g-f)K(k) + \frac{f^2-k^2}{f}\pi_o(k,-\frac{k^2}{f^2}) + \frac{k^2-g^2}{g}\pi_o(k,-\frac{k^2}{g^2})\right] \quad \text{-----------(21)}$$

Eq.(19), upon the integration and simplification yields,

$$\frac{a}{b} = \mu\cos\beta + \frac{1-\mu^2}{\pi}\tan^{-1}\frac{2\mu}{1-\mu^2} - \frac{\mu\sin\beta}{\pi}\ln\frac{1+\sin\beta}{1-\sin\beta} \quad \text{-----------------------(22)}$$

Eqs.(21) and (22) are the explicit forms of the desired expressions, X/b and a/b as functions of the characteristic of the jet, C_c and β.

The equation of c/b as a function of C_c and β can be obtained by combining Eqs.(1), (13) and (16), thus

$$dz = -\frac{\mu b}{\pi k}\left(\frac{1}{t+f} - \frac{1}{t+g}\right)\frac{\sqrt{t^2-k^2} - t\sqrt{1-k^2}}{\sqrt{t^2-1}}\,dt \quad \text{---------------------(23)}$$

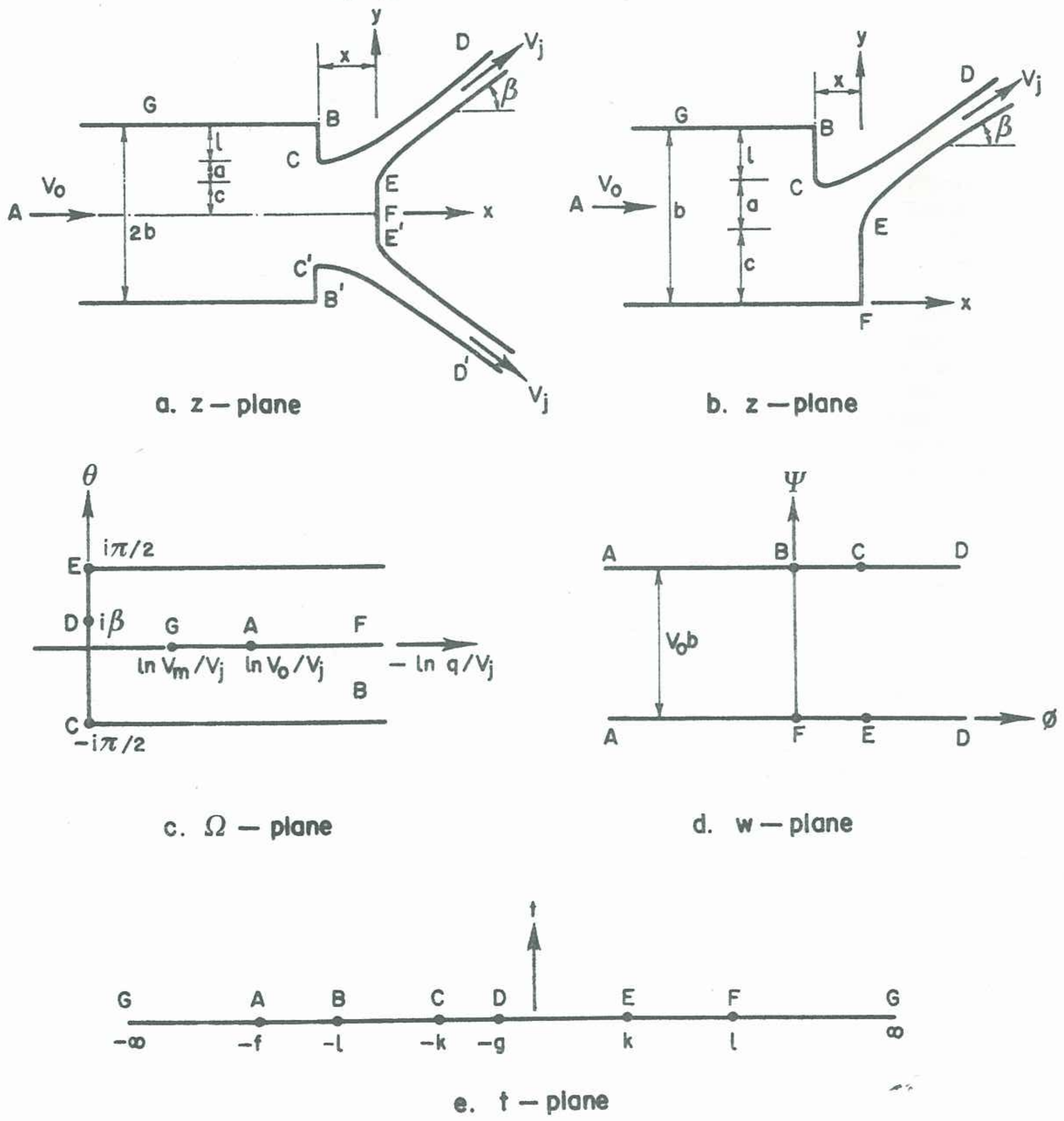


Fig.1 Flow Through a Two-dimensional Slot Past a Rectangular Plate Placed Near the End of the Slot

Integrating Eq.(23) along the boundary FE yields

$$c = \frac{\mu b}{\pi k}\left[\int_{-1}^{-k}\left(\frac{1}{t+f}-\frac{1}{t+g}\right)\frac{\sqrt{t^2-k^2}}{\sqrt{1-t^2}}\,dt - \int_{-1}^{-k}\left(\frac{1}{t+f}-\frac{1}{t+g}\right)\frac{t\sqrt{1-k^2}}{\sqrt{1-t^2}}\,dt\right] \quad \text{------------------(24)}$$

which after integration and simplification becomes

$$\frac{c}{b} = \frac{1}{2}(1-\mu\cos\beta) - \frac{\mu}{\pi k}(f-g)K(\sqrt{1-k^2}) + \frac{1-\mu^2}{2\pi}\tan^{-1}\frac{1-fk}{\sqrt{(f^2-1)(1-k^2)}} + \frac{\mu f(f^2-k^2)}{\pi k(f^2-1)}\pi_o\left(\sqrt{1-k^2}, \frac{1-k^2}{f^2-1}\right)$$

$$+ \frac{\mu\sin\beta}{\pi}\ell n\frac{\sqrt{1-k^2\cos^2\theta}+1}{k(1-\sin\beta)} - \mu gk\cos^2\beta\,\pi_o(\sqrt{1-k^2}, k^2\cos^2\beta-1) \quad \text{------------------(25)}$$

Given the geometry of an approach channel viz: e/b, a/b and X/b, the values of C_c and β can be obtained by solving the six equations (12,(15),(21),(22),(25) and the continuity equation $\mu = V_o/V_j = C_c a/b$. The complete eliptic integral of the first and third kinds were evaluated by the computer programs from Wang (6). The solution was numerically determined by using an IBM 1130 computer.

It is difficult to prepare plots showing C_c and β as functions of a/b, c/b and X/b. The numerical solutions are thus only given for the case X = 0, which are shown in Figs.3 and 4. In Figs.3 and 4, the dotted lines represent the behaviors of the limiting cases of a symmetrical slot ($c = \ell$) and a slot located on one side of the approach channel ($\ell = 0$).

VERIFICATION OF RESULTS

Chowchuvech (3) experimentally investigate the limiting case of the problem when X = 0. The experimental values of C_c and β from Chowchuvech agreed well with the theoretical values as shown in Figs.3 and 4. Figs.5 and 6 show the relationships between the theoretical and experimental values of C_c and β respectively.

The experimental values of the coefficient of contraction are however always above the theoretical line. The authors suspect that the reasons for this discrepancy of the coefficient of contraction is mainly due to the use of potential flow theory. In real fluid the pattern of flow is influenced by viscosity. This effect causes the experimental value of the velocity of the jet to be smaller than that of the theoretical solution. The area of the jet was determined by Chowchuvech to be equal to the ratio between the discharge of the flow and the velocity of the jet. Hence, the experimental C_c is always higher than the theoretical values.

CONCLUSIONS

In solving problems of this type, care should be made in considering the point where the maximum velocity occurs along the boundary of the flow especially when this point becomes a vortex of a polygon in an intermediate plane.

The technique of integration across free streamlines played an important role in simplifying the analysis of a two-dimensional free-jet problem. This technique was used to modfy Zhukovikii's method in obtaining the parametric equations. It offers a way to obtain the parametric equations directly by equating the real and imaginary parts of the differential equation before carrying out the integrations.

The theoretical solutions of the problem using potential flow theory provided close agreement to the experimental results obtained using a viscous incompressible fluid.

REFERENCES

(1) Arbhabhirama, A., "Theoretical Study of the Flow and Discharge Characteristics of a Two-dimensional Orifice (Slot) Placed Unsymmetrically in the Approach Channel", Thesis No.18, SEATO Graduate School of Engineering, Bangkok, Thailand, 1962.

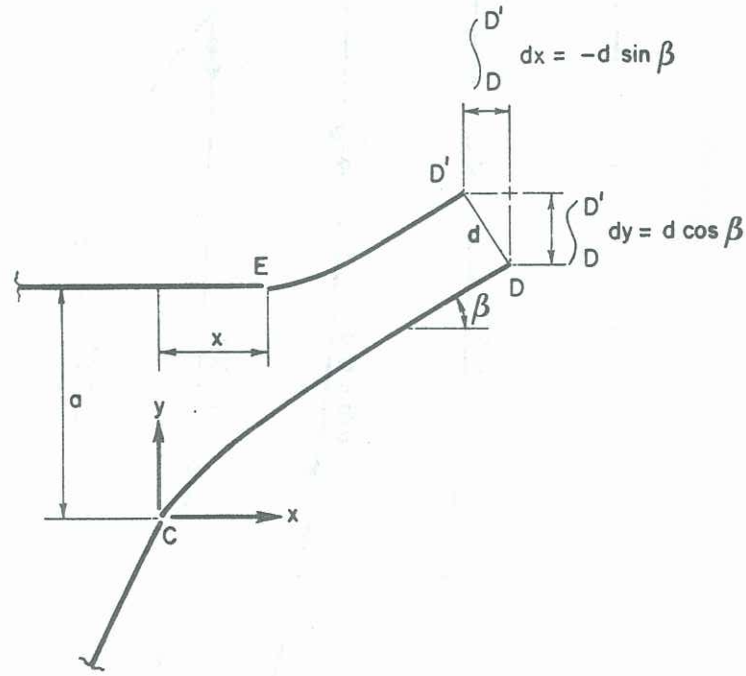
(2) Arbhabhirama, A., "Free Streamlines Analysis of Two-dimensional Jet", Journal of the Hydraulic Division, ASCE, Vol.95, Proc. Paper 6658, July 1969, pp.1139-1148.

(3) Chowchuvech, S., "Flow and Discharge Characteristics of a Two-dimensional Orifice (Slot) Placed Unsymmetrically in the Approach Channel", Thesis No.2, SEATO Graduate School of Engineering, Bangkok, Thailand, 1961.

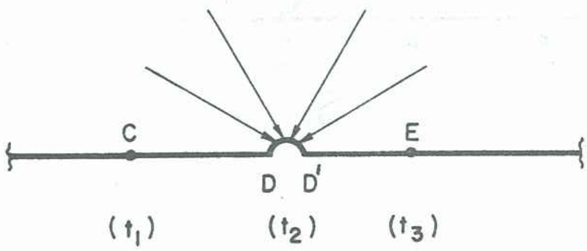
(4) Gurevich, M.I., "Theory of Jets in Ideal Fluids", Academic Press, New York, 1965.

(5) von Mises, R., "Berechnung von Ausfluss und Veberfallzahlem, Zeits", VDI 61, pp.447-452, 467-473, 493-497, 1961.

(6) Wang, T.H., "Characteristics of the Jet of Two-dimensional Orifice Past a Rectangular Plate", Thesis No.308, Asian Institute of Technology, Bangkok, Thailand, 1970.



a. Flow in z-plane



b. Flow in t-plane

Fig.2 Flow Pattern of Free Jet Discharging from Slot

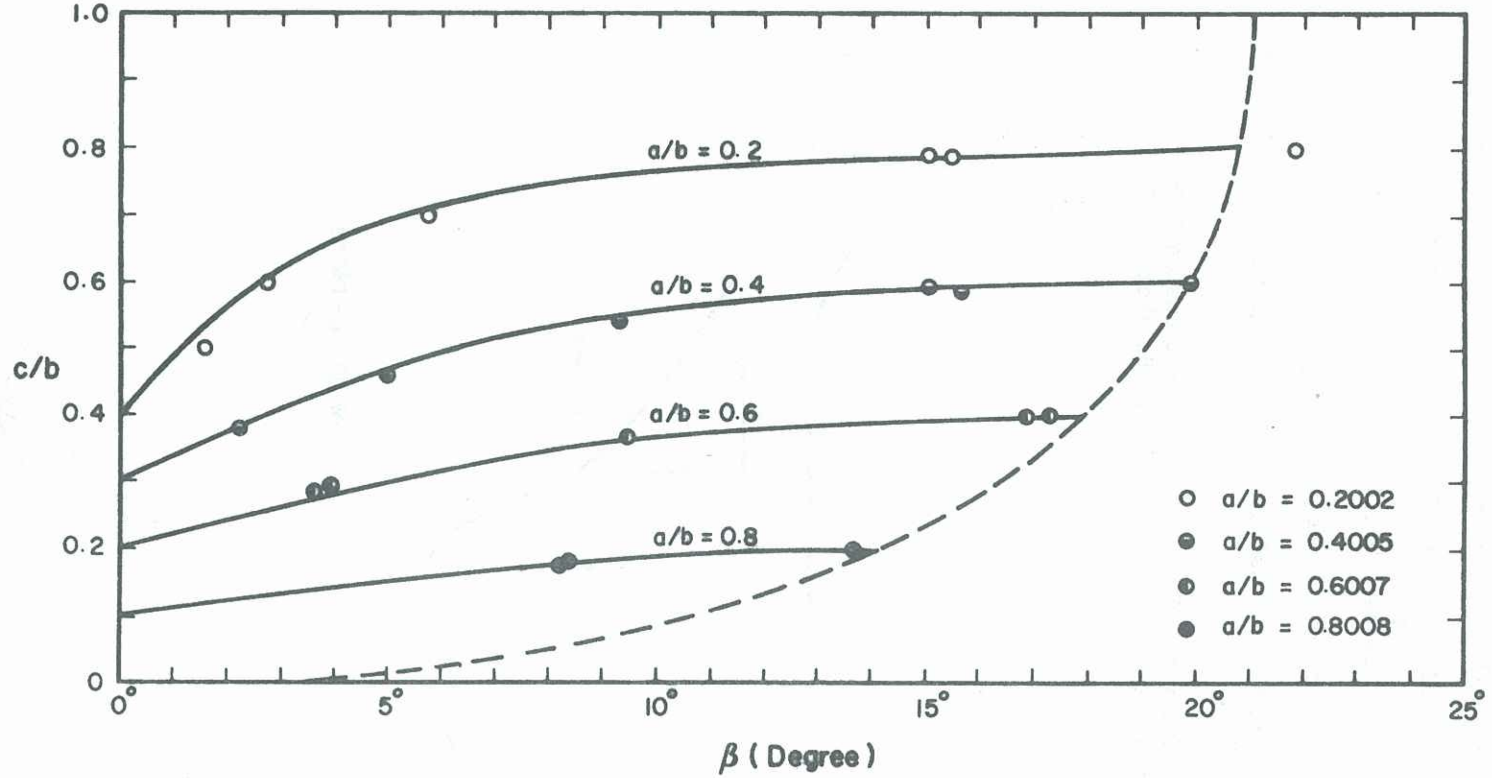


Fig. 4 Angle of Deflection as a Function of the Geometry of the Channel for x = 0

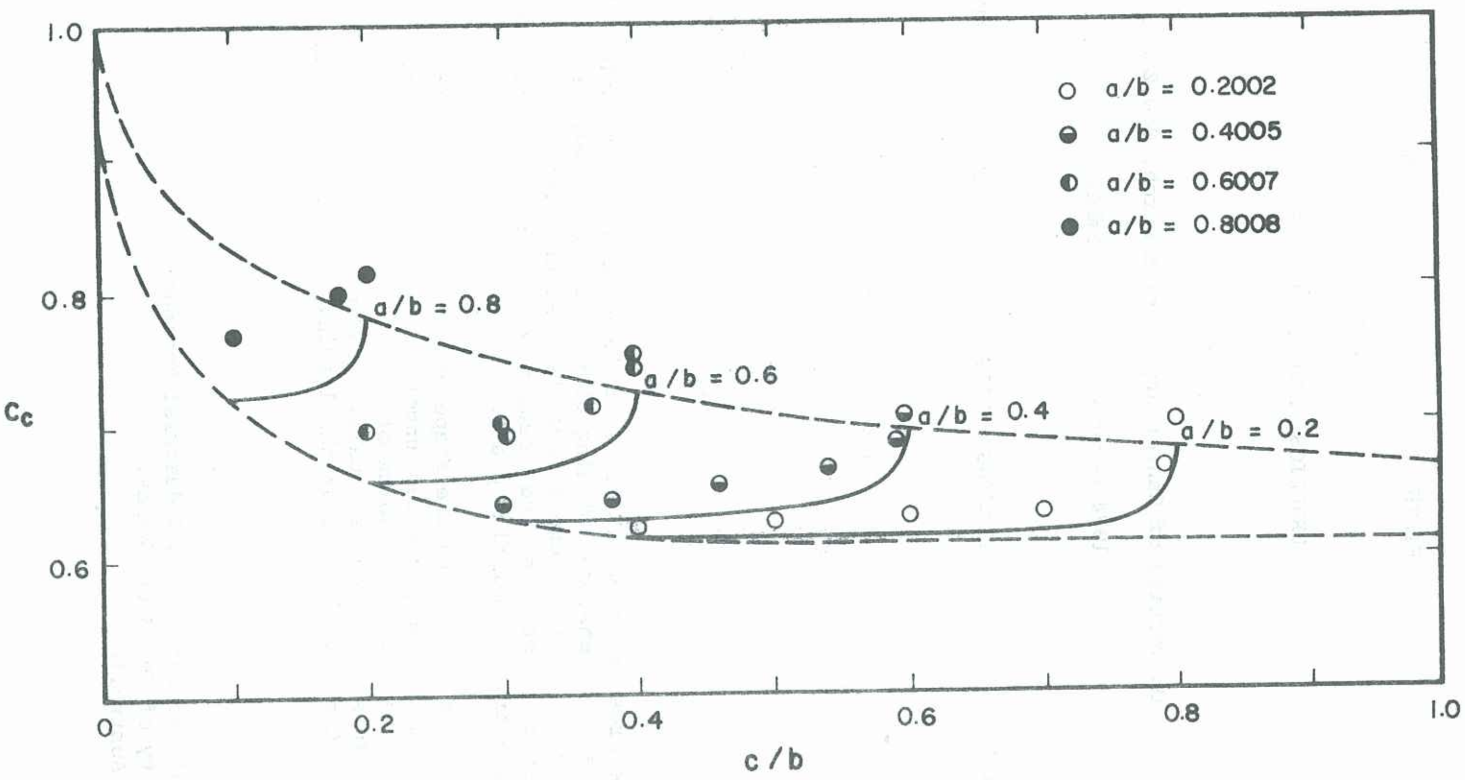


Fig. 3 Coefficient of Contraction as a Function of the Geometry of the Channel for x = O