FIFTH AUSTRALASIAN CONFERENCE

on

HYDRAULICS AND FLUID MECHANICS

at

University of Canterbury, Christchurch, New Zealand

1974 December 9 to December 13

POTENTIAL FLOW THROUGH A CONTRACTED SECTION

by

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SUMMARY

An approximate potential flow solution is obtained for the discharge through two dimensional T-shaped contractions. The analytic solution is obtained using methods of conformal mapping. In spite of the apparently simple geometric configuration of the boundary conditions, exact solutions are not available because of the inability to evaluate in closed form sets of incomplete elliptic integrals of the third kind. The approximate solutions which are presented require the use of several diagrams which have been prepared to facilitate the use of the equations describing the flow through the T-shaped sections. These results were verified experimentally using tests conducted on an electric analog model.

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INTRODUCTION

Two-dimensional potential flow through the contracted section shown in Fig.1 was investigated. In spite of the apparently simple geometric configuration of the boundaries which contain the flow, no exact solutions are known to the authors, which describe the conditions resulting from differing potentials located at finite distances upstream and downstream from the contractions.

The approximate solution to these two problems is described in terms of conditions produced by flow through a porous media. The solution would be equally appropriate for any field problems which satisfy the Laplace equation such as the flow of heat through a plate where the impervious by surfaces of constant temperature.

MATHEMATICAL ANALYSIS

Boundary Conditions

Consider a steady, two-dimensional flow in a homogeneous, isotropic porous media shown in Fig.1. The impervious boundaries AFED and BA-FED for cases A and B respectively are streamlines defined by the stream function ψ having a magnitude equal to zero. The central vertical line BC of case A and the impervious boundary AB and central vertical line BC of case B are streamlines defined by the stream function ψ having a magnitude -q equal to the negative of the discharge. The heads at inlet and outlet for both cases are H and h respectively.

Method of Solution and Definition of Complex Planes

To determine the discharge through the two dimensional shapes shown in Fig.1 methods of conformal mapping were used. Unfortunately, an exact solution for the case where L_2 was finite was not obtained because of difficulties in the integration of hyperelliptic integrals. An approximate solution for this condition was however obtained. As the length L_2 increases, other dimensions remaining fixed, the flow in the downstream part of reach L_2 approaches uniform flow.

The solution for the case where the distance L shown in Figs.2 and 5 is finite is approximated using the solution for conditions when L_2 is infinite. This is accomplished by determining the location along the contracted section of infinite length where the flow approaches uniform conditions. This condition was considered to occur at that point where the centerline velocity along the contracted section was at least 99% of the velocity which exists for conditions of uniform flow. Downstream from this 99% section uniform flow was considered to exist for the entire reach to the end of the contracted section.

Relationship between Complex Planes: Case A Using the Schwartz-Christoffel Theorem a relationship was obtained between the z and p planes. Substituting $p = t^2$ and using the boundary conditions at points B and A shown in Fig.2, the relationship between z and t was obtained as

$$z = i \frac{\Pi(m,n,t)}{\Pi_o(m,n)} (\ell_1 + \ell_2) \tag{1}$$

where $\Pi(m,n,t)$ and $\Pi_o(m,n)$ are the incomplete and complete elliptic integrals of the third kind with modulus m, parameter n and amplitude t, Byrd and Friedman (1).

Similarly, the relationship between the w and t planes was obtained as

$$w = \frac{q}{\pi} \ell n \frac{\sqrt{t^2-1} + t\sqrt{1+n}}{\sqrt{t^2-1} - t\sqrt{1+n}} - kH - iq \tag{2}$$

Determination of L_1 and ℓ_2 - Substituting the boundary conditions at points F and E into Eq.(1), the following equations were obtained

$$L_1' = \frac{m^2}{m^2+n} \frac{\Pi_o(m',n')}{\Pi_o(m,n)} (1+\ell_2') \tag{3}$$

$$\ell_2' = \frac{2}{\pi} \Pi_o(m,n) \sqrt{(1+n)(1+m^2/n)} - 1 \tag{4}$$

where the dimensionless lengths are defined as $L_1' = L_1/\ell_1$, $\ell_2' = \ell_2/\ell_1$ and the modulus and parameters m' and n' are defined as $m' = \sqrt{1-m^2}$ and $n' = -\frac{nm'^2}{m^2+n}$.

Determination of length L - The velocity in the x-direction at a point J located a distance L along BC from the sudden contraction, as shown in Fig.1 case A, is expressed as

$$(V_x)_J = (\frac{\partial\phi}{\partial x})_J = (\frac{\partial\phi}{\partial t}\frac{\partial t}{\partial x})_J = -\frac{\varepsilon q}{\ell_1} \qquad (5)$$

where the ratio of the local velocity to the velocity of uniform flow is equal to ε. The terms $(\frac{\partial t}{\partial x})_J$ and $(\frac{\partial\phi}{\partial t})_J$ can be obtained by taking the derivative of Eqs.(1) and (2) with respect to t respectively and substituting the boundary conditions at point J. Eq.(5) then becomes

$$\frac{2\Pi_o(m,n)\sqrt{1+n}}{\pi(1+\ell_2')}\sqrt{1-m^2t_J^2} = -\varepsilon \qquad (6)$$

In Eq.(6) t_J is the coordinate at the point J on the t plane and at this point the velocity of flow is $\varepsilon q/\ell_1$. Substituting Eq.(4) into Eq.(6)

$$t_J = -\frac{1}{m}\sqrt{1-\varepsilon^2(1+m^2/n)} \qquad (7)$$

The length L can be obtained by substituting t_J and the boundary condition at point J where y = 0 into Eq.(1) to obtain

$$L' = A_1 - L_1' \qquad (8)$$

where $L' = L/\ell_1;\quad L' = L_1/\ell_1$

and $A_1 = i\frac{2}{\pi}\Pi(m,n,t_J)\sqrt{(1+n)(1+m^2/n)} \qquad (9)$

Determination of discharge when L_2 is finite - The velocity potential ϕ_J across section GJ in Fig.1 can be approximately by substituting t_J into Eq.(2) and taking the real part of the resulting expression

$$J = \frac{q}{\pi}\ell n\frac{\sqrt{t_J^2-1}+t_J\sqrt{1+n}}{\sqrt{t_J^2-1}-t_J\sqrt{1+n}} - kH \qquad (10)$$

If ε approaches unity, flow after section GJ can be considered as uniform flow. Therefore the discharge through the contracted section can be approximated by applying Darcy's law in the reach downstream of section GJ which has a length L_2-L.

$$q = -\frac{\phi_J + kH}{L_2-L}\ell_1 = -\frac{\phi_J + kH}{L_2'-L'} \qquad (11)$$

In Eq.(11) $L_2' = L_2/\ell_1$. Substituting Eqs.(8) and (10) into (11), the equation for the discharge q was obtained as

$$q = \frac{k(H-h)}{L_1' + L_2' - A_1 + A_2} \qquad (12)$$

where

$$A_2 = \frac{1}{\pi}\ell n\frac{\sqrt{1-1/t_J^2}+\sqrt{1+n}}{\sqrt{1-1/t_J^2}-\sqrt{1+n}} \qquad (13)$$

In order to use Eq.(13) to determine the discharge, A_1 and A_2 must be known. All physical dimensions are assumed to be given in the definition of the problem. Using Eqs.(3) and (4), Fig.3 was prepared which gives the relationship between ℓ_2', L_1', m and n. Since ℓ_2' and L_1' are known from the geometry of the problem the values of m and n can be determined. Employing Eqs.(9) and (13), Fig.4 was prepared which describes the relationship between A_1, A_2, m and n. In Fig.4 the value of ε is 0.998 which means that the velocity along the boundary at point J is 0.2% less than the velocity of uniform flow which occurs at an infinite distance downstream. To apply this analysis, the dimensionless length L_2' cannot be less than L' as obtained from Eq.(9).

Relationship between Complex Planes: Case B   Similar to case A the relationship between z and t for case B, as shown in Fig.5, was obtained as follows:

$$z = i \frac{\Pi(m,n,t)}{\Pi_o(m,n)} (\ell_1 + \ell_2) \tag{14}$$

Similarly, the relationship between the w and t planes was obtained as

$$w = \frac{q}{\pi} \ln \frac{\sqrt{(m^2t^2-1)(1+n)} + \sqrt{(t^2-1)(n+m^2)}}{\sqrt{(m^2t^2-1)(1+n)} - \sqrt{(t^2-1)(n+m^2)}} - kH - iq \tag{15}$$

Determination of L_1, ℓ_2 and L - Using the same procedure as in case A, the following equations were obtained

$$L_1' = \frac{m^2}{m^2+n} \frac{\Pi_o(m',n')}{\Pi_o(m,n)} (1+\ell_2') \tag{16}$$

$$\ell_2' = \frac{2}{\pi} \Pi_o(m,n) \sqrt{(1+n)(1+m^2/n)} - 1 \tag{17}$$

where $L_1' = L_1/\ell_1$ and $\ell_2' = \ell_2/\ell_1$.

$$t_J = -\frac{\varepsilon i}{\sqrt{n}} \tag{18}$$

$$L' = B_1 - L_1' \tag{19}$$

$$B_1 = i \frac{2}{\pi} \Pi(m,n,t_J) \sqrt{(1+n)(1+m^2/n)} - L_1' \tag{20}$$

Determination of discharge when L_2 is finite - Substituting t_J into Eq.(15) and solving for ϕ_J at section GJ shown in Fig.1 case B,

$$\phi_J = \frac{q}{\pi} \ln \frac{\sqrt{(m^2t_J^2-1)(1+n)} + \sqrt{(t_J^2-1)(n+m^2)}}{\sqrt{(m^2t_J^2-1)(1+n)} - \sqrt{(t_J^2-1)(n+m^2)}} - kH \tag{21}$$

At point J where ε approaches unity, conditions of uniform flow may be assumed to exist at sections further downstream. By applying Darcy's law in the uniform flow reach L_2-L the discharge across the contraction was obtained as

$$q = -\frac{\phi_J + kh}{L_2-L} \ell_1 = \frac{\phi_J + kh}{L_2' - L'} \tag{22}$$

where $L_2' = L_2/\ell_1$. Substituting Eqs.(19) and (21) into (22).

$$q = \frac{k(H-h)}{L_1'+L_2'-B_1+B_2} \tag{23}$$

where

$$B_2 = \frac{1}{\pi} \ln \frac{\sqrt{1+n} + \sqrt{(n+m^2)(t_J^2-1)/(m^2t_J^2-1)}}{\sqrt{1+n} - \sqrt{(n+m^2)(t_J^2-1)/(m^2t_J^2-1)}} \tag{24}$$

The relationship between ℓ_2', L_1', m and n in Eqs.(16) and (17) are shown in Fig.3. Using Eqs.(20) and (24), Fig.6 was prepared which gives the relationship between B_1,B_2, m and n. In Fig.6 the value of ε is 0.990 which means that the ratio of the local to the velocity of uniform flow is equal to 0.990 at point J. To apply this analysis, the dimensionless length L_2' cannot be less than L' as obtained from Eq.(19).

PRESENTATION OF RESULTS

The relationships between ℓ_2', L_1', L_2', A_1, A_2, B_1 and B_2 shown in the equations are dependent upon the modulus m, parameter n, and ε. It is seen that a direct solution of these problems in which the geometry of the flow region is considered as the independent variable is not practical. Numerical solutions for a range of physical configurations were therefore computed and the results presented in graphical form as shown in Figs.3, 4 and 6.

APPLICATION

Consider the flow in regions geometrically similar to those in Fig.1 which have the following characteristics: permeability of the media k = 0.002 ft/sec, H-h = 15 ft, $\ell_2' = \ell_2/\ell_1 = 1$, $L_1' = L_1/\ell_1 = 1.42$ and $L_2' = L_2/\ell_1 = 2$.

From Fig.3 the values of m and n for both cases A and B is determined as m = 0.80 and n = 0.40. From Figs.4 and 6 the values of A_1, A_2, B_1 and B_2, required in Eqs.(12) and (23), are determined as $A_1 = 2.61$, $B_1 = 2.55$, $A_2 = 2.15$, and $B_2 = 2.42$.

Therefore, the discharge and the length L' for cases A and B are determined from Eqs.(12) and (23) as

Case A

$$\frac{q}{k(H-h)} = \frac{1}{L_1'+L_2'-A_1+A_2} = 0.338 \qquad (25)$$

For the results of Eq.(25) to apply the following inequality must also be satisfied

$$L' = A_1 - L_1' = 1.19 < L_2' \qquad (26)$$

Case B

$$\frac{q}{k(H-h)} = \frac{1}{L_1'+L_2'-B_1+B_2} = 0.304 \qquad (27)$$

For the results of Eq.(27) to apply the following inequality must also be satisfied

$$L' = B_1 - L_1' = 1.13 < L_2' \qquad (28)$$

ELECTRIC ANALOG MODEL

Confirmation of the accuracy of the analytic analysis was obtained from tests conducted using an electric analog model. The problems described in the sample calculations were used for the experimental models. The experimentally determined flow nets are shown in Chang (2). The number of equipotential drops N_e and the number of flow channels N_f for cases A and B described in the Applications were obtained. The dimensionless discharges were computed as follows:

Case A

$$\frac{q}{k(H-h)} = \frac{N_f}{N_e} = \frac{6.6}{20} = 0.330$$

Case B

$$\frac{q}{k(H-h)} = \frac{N_f}{N_e} = \frac{5.85}{20} = 0.293$$

Comparing these with the theoretically obtained values, the differences for cases A and B are 2.37% and 3.62% respectively.

CONCLUSIONS

An approximate solution of the two dimensional Laplace Equation is obtained for the flow rate through a contraction in which a plane of constant potential is specified at an up and downstream section. The solution involving terms containing elliptic integrals of the third kind is expressed in graphical form to facilitate an application of the resulting equations.

REFERENCES

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(2) Chang, Y.Y., Salt water-fresh water interface during groundwater pumping and equivalent single phase flow system, Thesis No.311, Asian Institute of Technology, Bangkok, Thailand.

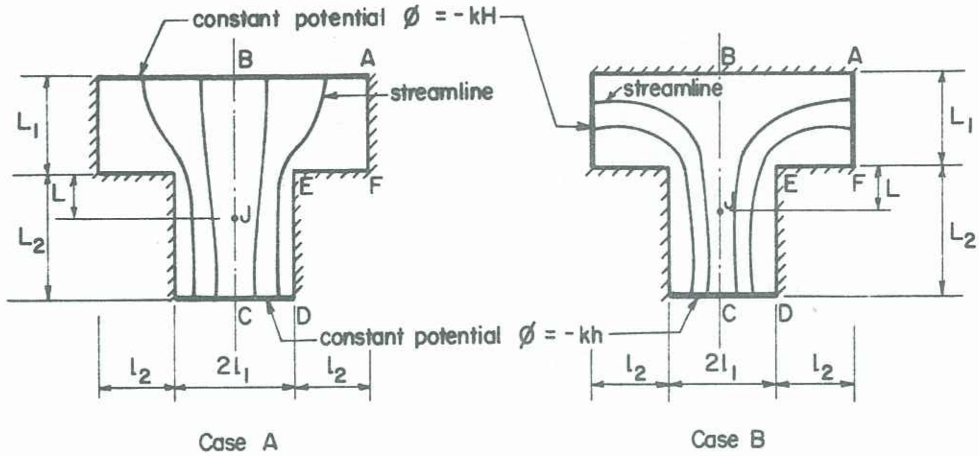

Fig. 1 Definition Sketch of Flow Through a Contraction

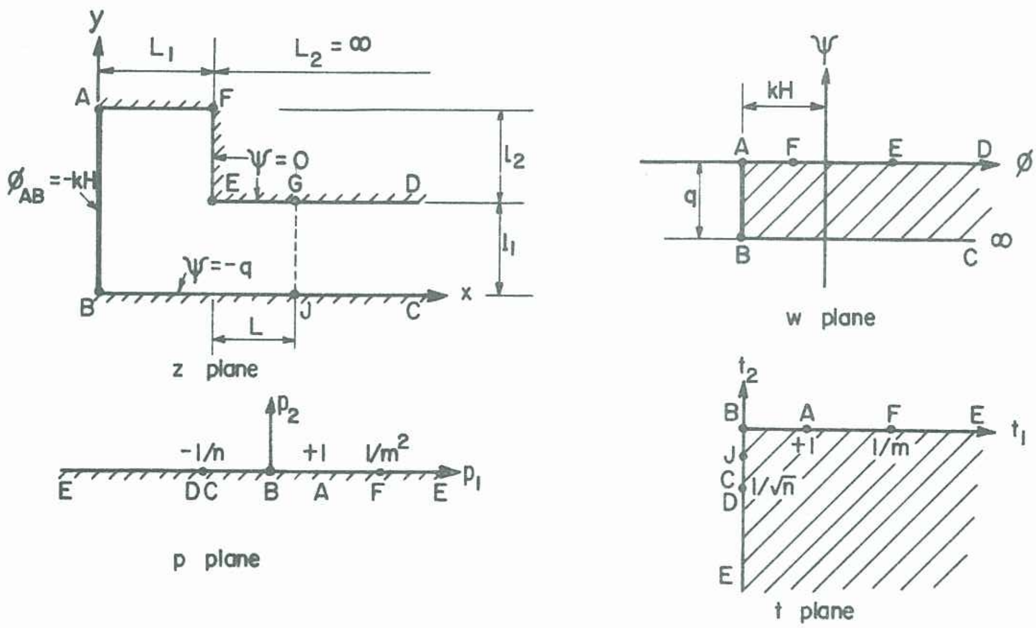

Fig. 2 Complex Planes : Case A

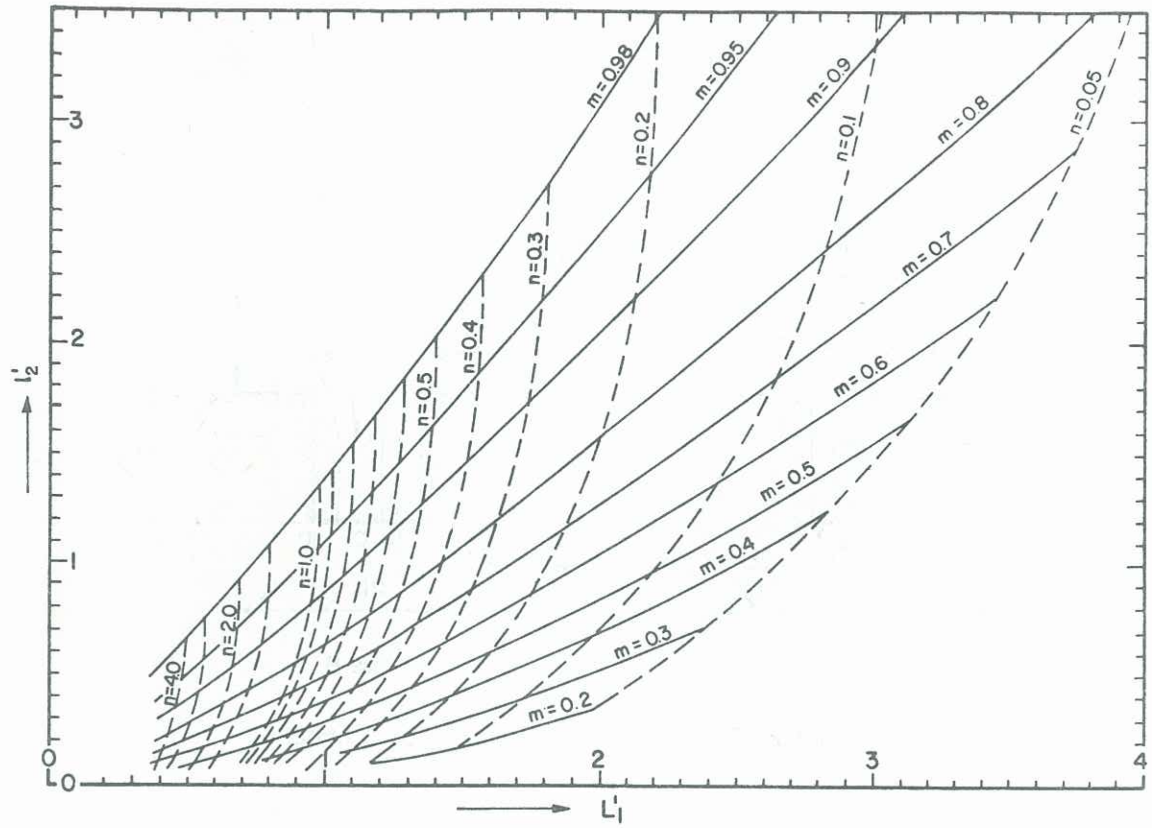

Fig. 3 Relationship between m, n, l'_2 and L'_1

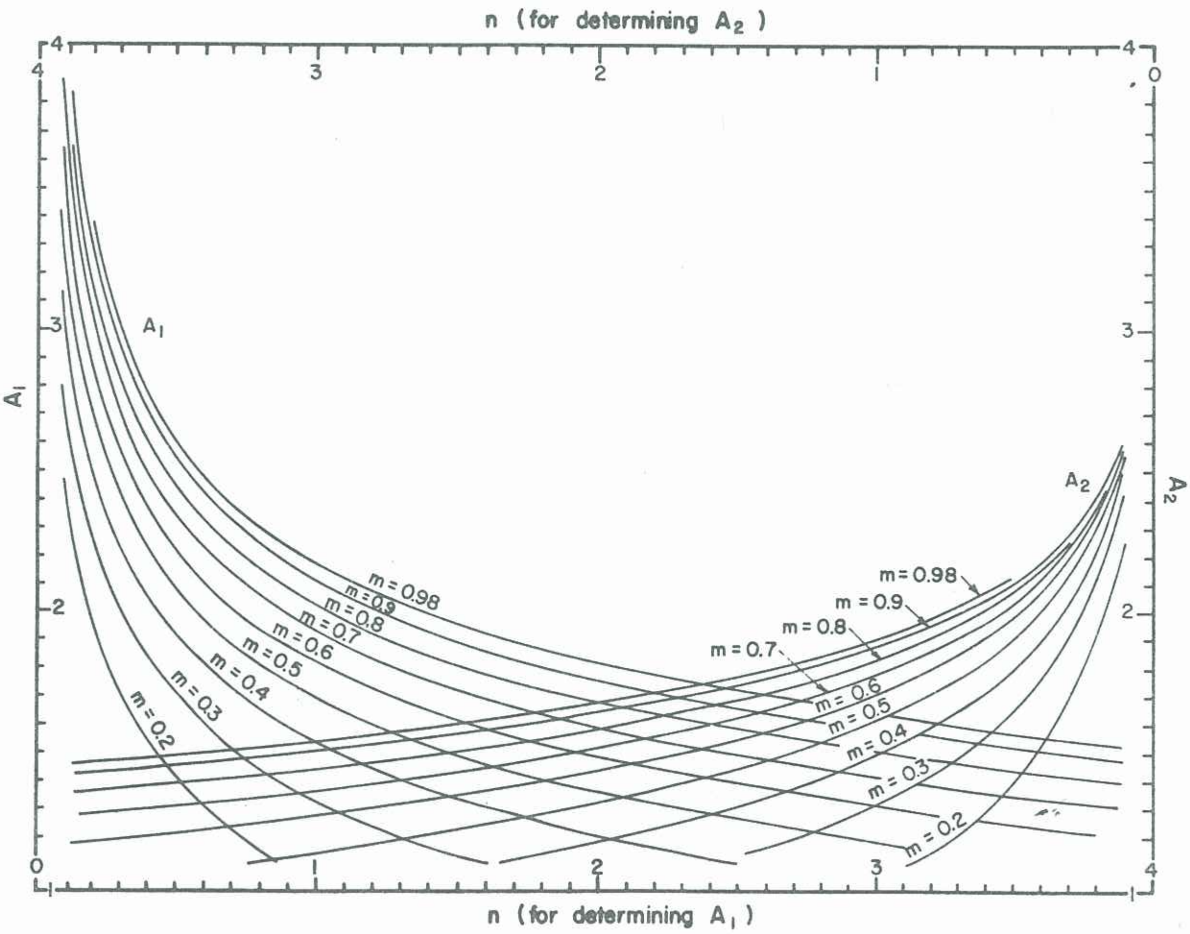

Fig. 4 Relationship between m, n, A_1 and A_2

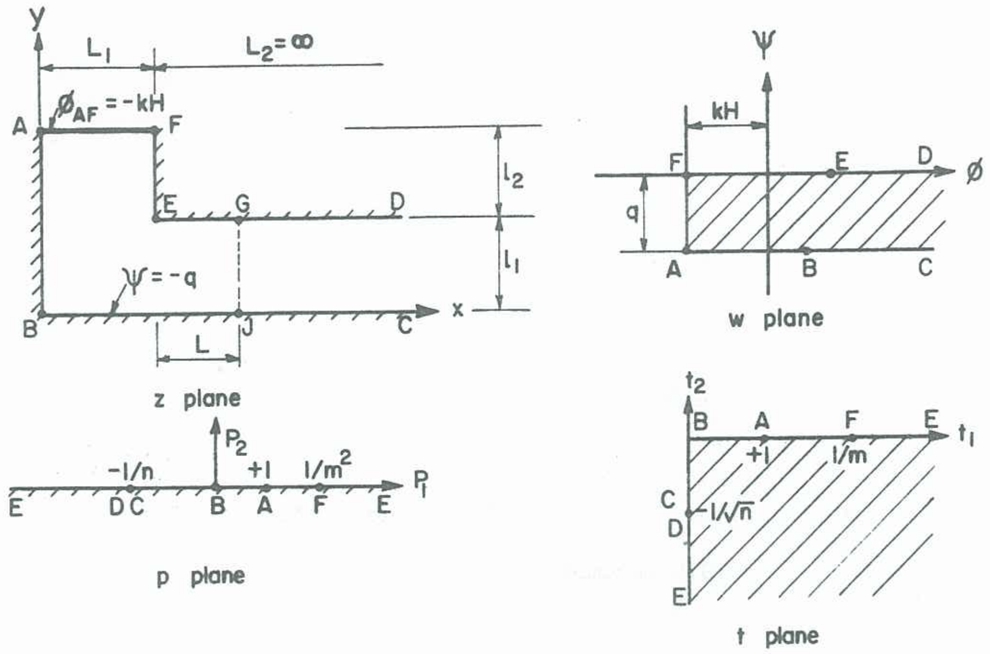

Fig. 5 Complex Planes : Case B

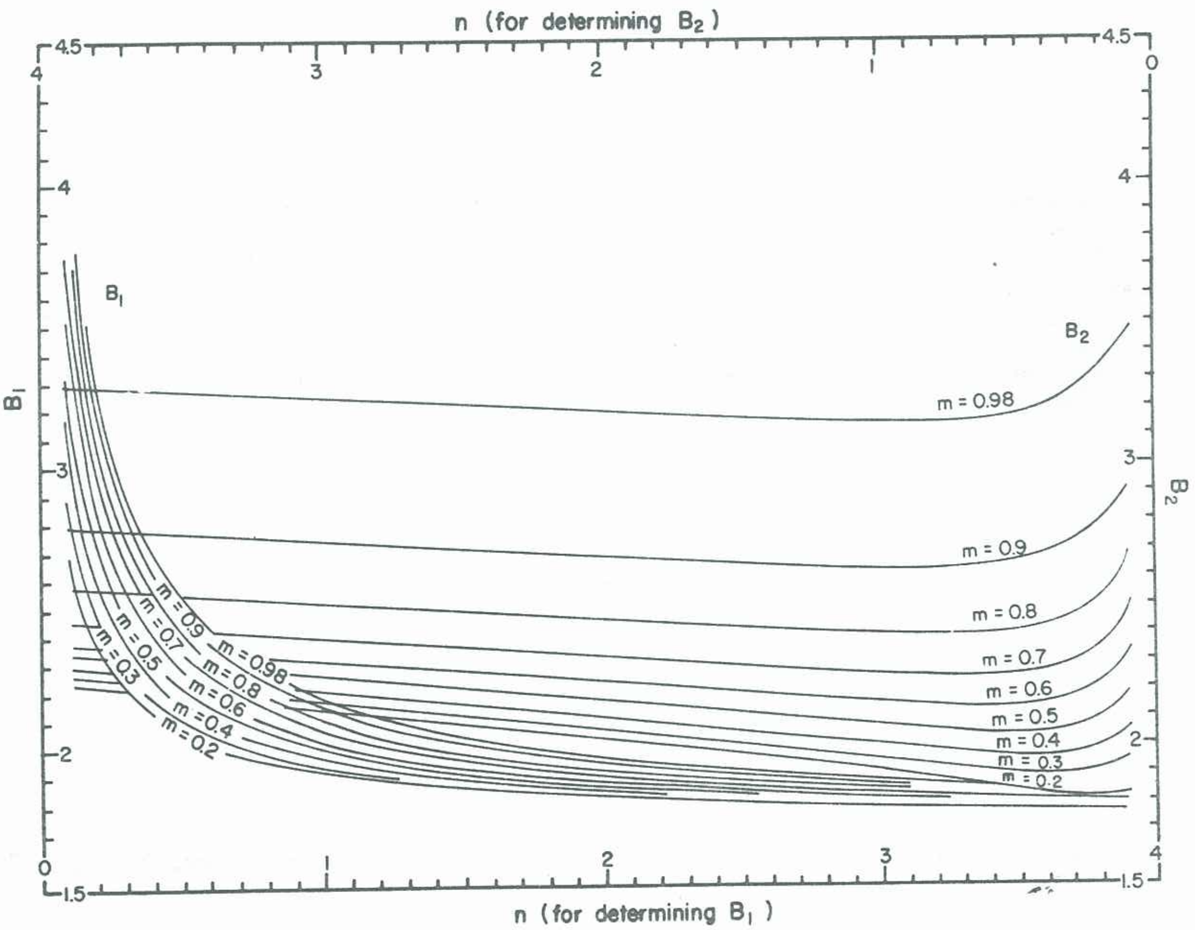

Fig. 6 Relationship between m , n , B_1 and B_2