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ACCELERATED MOTION OF A SPHERE IN LAYERED FLUIDS

by

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SUMMARY

The various forces involved in the accelerated motion of a body in a viscous fluid are examined. The equation of motion for a sphere, in particular, is discussed. Accelerated motion of a sphere in layered fluids is analysed and solved using numerical integration technique. General solution procedure for the motion of sphere in multilayer fluids is indicated and some particular cases are analysed.

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INTRODUCTION

The fall velocity of particles in a viscous fluid is an important aspect in the problems of Sediment transportation, Mineral seperation, Gravimetric analysis and other related disciplines. Stokes (1), Basset (1), Brush et al (2) Hjelmfelt et al (4) etc. have reported the formulation and solution of this problem. The fall velocity of a spherical body in a layered fluid medium is of more significance in dealing with the density current deposits in reservoirs. In such cases the density and viscosity of water vary with depth.

In this investigation the accelerated motion of a spherical particle in a layered viscous fluid is analysed with a view that this might throw some light on the mechanics of the actual sedimentation problem that takes place in large and deep reservoirs and rivers. In classical studies, usually, the steady state velocity given by Stokes, is generally used, thus neglecting the accelerated motion of the spherical particle before it reaches the terminal velocity.

The basic studies on the accelerated motion of a sphere in a homogeneous viscous fluid were due to Basset (1), who presented the solution neglecting convective acceleration terms of the Navier-Stokes equation. Brush et al (2) and Hjelmfelt et al (4) have presented their solutions in two different ranges of density ratios.

In this investigation, the various forces involved in the accelerated motion of a sphere in a viscous fluid are considered. Accelerated motion of a sphere in layered fluid medium is analyzed using Runge-Kutta fourth order numerical integration technique, taking the velocity of the spherical body at the end of a layer to be equal to the initial velocity of the sphere for the next layer. Thus, the solution procedure boils down to solving the problem of a sphere with and without initial velocity in any layer. A program is prepared for a 'n' layer case and a three layer problem, in particular, is presented graphically with different non-dimensional viscosity and density ratios.

ANALYSIS

The accelerated motion of a sphere in a homogeneous viscous fluid has been derived by Basset(1), assuming the resistance to the sphere motion to be a linear function of the velocity and neglecting higher order velocity terms. Thus, the forces involved in such a case can be obtained as follows (3).

1. Buoyant force $= (m_s - m_f)\, g$ ... (1a)
2. Inertia force $= - m_s \dot{v}$ ... (1b)
3. Virtual mass effective force $= - k m_f \dot{v}$ ... (1c)
4. Dissipative forces on the sphere $= 3\pi\mu\, dv - \frac{3}{2} d^2 \sqrt{\pi \rho_f \mu} \int_0^t \frac{\frac{dv}{d\tau}}{\sqrt{t-\tau}}\, d\tau$

$0 \leq \tau \leq t$ ... (1d)

The displacement co-ordinate of the sphere is considered positive in the vertically down ward direction, and,

ρ_s = Density of the sphere material,
m_s = Mass of the sphere,
ρ_f = Density of the fluid medium
$R = \rho_s/\rho_f$ = Density ratio,
m_f = Mass of the volume of the fluid displaced by the sphere,
k = Added mass coefficient = 1/2 for the sphere,
μ = Viscosity of the fluid medium,
$\nu = \mu/\rho_f$ = Kinematic viscosity of the fluid,
g = Acceleration due to gravity,
d = Diameter of the sphere, and, v = Velocity of the sphere at any time, t.

The integral term in Eq. (1d) is a type of history term indicating that the resistance to the sphere motion at any time, t, due to unsteadiness of flow, is, in part, a function of the resistance to motion at a previous time. This accounts for the transient character of the velocity distribution. Thus, the equation of the force balance for the sphere motion can be written as;

$$m_s \frac{dv}{dt} + k\, m_f \frac{dv}{dt} = (m_s - m_f)\, g - 3\pi\mu d v - \frac{3}{2} d^2 \sqrt{\pi \rho_f \mu} \int_0^t \frac{\frac{dv}{d\tau}}{\sqrt{t-\tau}}\, d\tau \qquad \ldots (2)$$

Now, Eq. (2) has to be solved using two initial conditions as follows:

Case I: Initial Velocity is Zero

If the particle starts from rest from the position $x = 0$ at time $t=0$, and $v = dx/dt = 0$, using Abel's Transformation(3), the history integral can be evaluated as;

$$\frac{3}{2} d^2 \rho_f \sqrt{\pi \nu} \int_0^t \frac{\frac{dv}{d\tau}}{\sqrt{t-\tau}}\, d\tau = \frac{3}{4} d^2 \rho_f \cdot \frac{\pi}{\sqrt{t}} \cdot \sqrt{\pi \nu} \cdot v \qquad \text{if } v\big|_{t=0} = 0 \qquad \ldots (3)$$

Using Eq. (3), Eq. (2) can be written in the non-dimensional form as;

$$R_A \frac{dV}{dT} + (1 + .442/\sqrt{T})V = V_t \qquad \ldots (4)$$

where, $R_A = (R+k)/18.0$; $T = \nu t/d^2$; $V = (dv/\nu)$; V_t = Stokes terminal velocity $= (R-1)\, gd^3/18\nu^2$ ... (5)

Defining; $Y = V/V_t$, ... (6)

in terms of these non-dimensional parameters, Eq. (4) can be rewritten as;

$$\frac{dY}{dT} = \left[1 - (1 + .442/\sqrt{T})\, Y\right]/R_A \qquad \ldots \qquad \ldots (7)$$

Case II: Particle Starting with an initial velocity 'v_o'

If v_o is the initial velocity of the sphere at the surface of a fluid, the same can be expressed in the non-dimensional form as;

$$V_o = (vd/\nu V_t)_{t=t_1} \qquad \ldots (8)$$

where, V_t is the Stokes terminal velocity of the sphere in that fluid layer under consideration. Eq. (7) can now be written as;

$$\frac{d\bar{Y}}{dT} = \left[1 - (1 + .442/\sqrt{T})\, \bar{Y}\right]/R_A \qquad \ldots \qquad \ldots (9)$$

where $\bar{Y} = (V - V_o)/(V_t - V_o)$ ... ... (10)

Now, Eqs. (7) and (9) have to be solved numerically. A fourth order Runge-Kutta method will be used for solving Eqs. (7) and (9) and accordingly they are reduced to sets of two first order simultaneous differential equations.

Case I: Initial Velocity is Zero

$$\frac{dX}{dT} = Y \qquad \ldots \qquad \ldots \qquad \ldots (11)$$

$$\frac{dY}{dT} = \left[1 - (1 + .442/\sqrt{T})\, Y\right]/R_A \qquad \ldots \qquad \ldots \qquad \ldots (12)$$

in which, $X = x/d.V_t$ ... ... (13)

Eqs. (11) and (12) have to be solved with the initial condition

$$X\big|_{T=0} = 0 \qquad \ldots \qquad \ldots \qquad \ldots (14)$$

$$Y\big|_{T=0} = 0 \qquad \ldots \qquad \ldots \qquad \ldots (15)$$

The limiting value of $\left(\frac{dY}{dT}\right)_{T \to 0}$ can be obtaned as

$$\left.\frac{dY}{dT}\right|_{T \to 0} = 1/R_A \quad \ldots \quad \ldots \quad \ldots (16)$$

Case II: Initial Velocity is v_o

Eq. (9) in this case can be written as;

$$\frac{d\bar{X}}{dT} = \bar{Y} \quad \ldots \quad \ldots (17)$$

$$\frac{d\bar{Y}}{dT} = \left[1 - (1 + .442/\sqrt{T})\ \bar{Y}\right]/R_A \quad \ldots \quad \ldots (18)$$

where $\bar{X} = (x/d - T\ V_0)/(V_t - V_0)$ ... ...(19)

Eqs. (17) and (18) have to be solved with the initial conditions;

$$\left.\bar{X}\right|_{T = 0} = 0 \quad \ldots \quad \ldots \quad \ldots (20)$$

$$\left.\bar{Y}\right|_{T = 0} = 0 \quad \ldots \quad \ldots \quad \ldots (21)$$

The limiting value of $\left.\left(\frac{d\bar{Y}}{dT}\right)\right|_{T \to 0}$ can be obtained as;

$$\left.\frac{d\bar{Y}}{dT}\right|_{T \to 0} = 1/R_A \quad \ldots \quad \ldots \quad \ldots (22)$$

Now, for this case (V/V_t) and $(x/d\ V_t)$, the non-dimensional values of velocity and displacement can be obtained, knowing the values of T at any stage from Eqs. (10) and (19) as:

$$(V/V_t) = V_o + (1 - V_o)\ \bar{Y} \quad \ldots \quad \ldots (23)$$

$$(x/d\ V_t) = V_o T + (1 - V_o)\ \bar{X} \quad \ldots \quad \ldots (24)$$

Motion of the sphere in Layered Fluid

If the sphere starts with zero initial velocity in the first layer, the displacement - time relationships and the velocity-time relationships can be obtained using Eqs. (11) to (16) of Case I. If the initial velocity is v_o in the first layer, the solutions can be obtained using Eqs. (17) to (22) of Case II. As soon as the sphere reaches the interface between any two layers, the time-displacement-velocity relationships can be obtained using Eqs. (16) to (21) of Case II with the initial velocity in the second layer being equal to the final velocity of the sphere in the first layer. Now, considering a 'n' layered fluid system, the procedure can be continued using equations of either Case I or Case II, as the case may be, with the initial velocity of the sphere in ith layer ($i \leq n$) being equal to the final velocity of the sphere in (i - 1)st layer, the displacement being compatible.

RESULTS AND DISCUSSIONS

A 3-layered medium is considered for illustrating the solution procedure with the following parameters:

1st layer: $R_1 = \rho_s/\rho_{f_1} = 10$: ν_1

2nd layer: $R_2 = \rho_s/\rho_{f_2} = 5$: $\nu_2/\nu_1 = 1.2$

3rd layer: $R_3 = \rho_s/\rho_{f_3} = 2.5$: $\nu_3/\nu_2 = 1.4$

The results are presented in terms of the non-dimensional parameters (expressing these parameters in terms of the properties of the fluid of the first layer). The non-dimensional final velocity $(V/V_t)_i$ of the top ith layer can be converted into the non-dimensional initial velocity of the (i+1)st layer, by multiplying $(V/V_t)_i$ with $\left[(R_i - 1)\,\nu_{i+1}/(R_{i+1}-1)\,\nu_i\right]$.

The solutions for a 3-layer case using the parameters given earlier are shown graphically in Figrs. (1) and (2). These can, in general, be extended for a wide range of the problems of practical significance.

CONCLUSIONS

A simple numerical technique has been presented to analyze the accelerated motion of a sphere in layered fluids. The usefulness of this technique has been tested by comparing the results obtained using this numerical procedure with some existing results elsewhere (3,5). It is expected that this technique can be successfully applied for more complex situations also.

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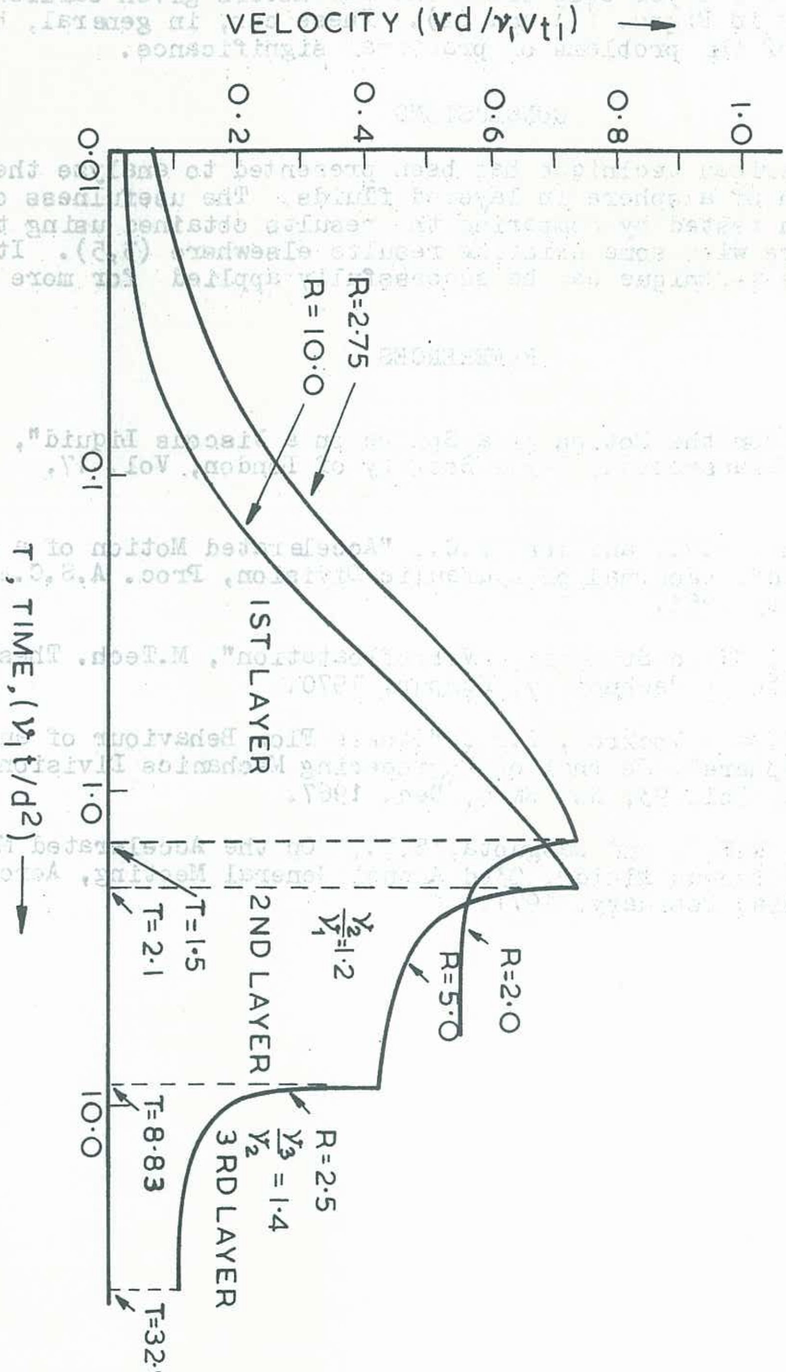

FIG.1, VELOCITY-TIME RELATION SHIPS

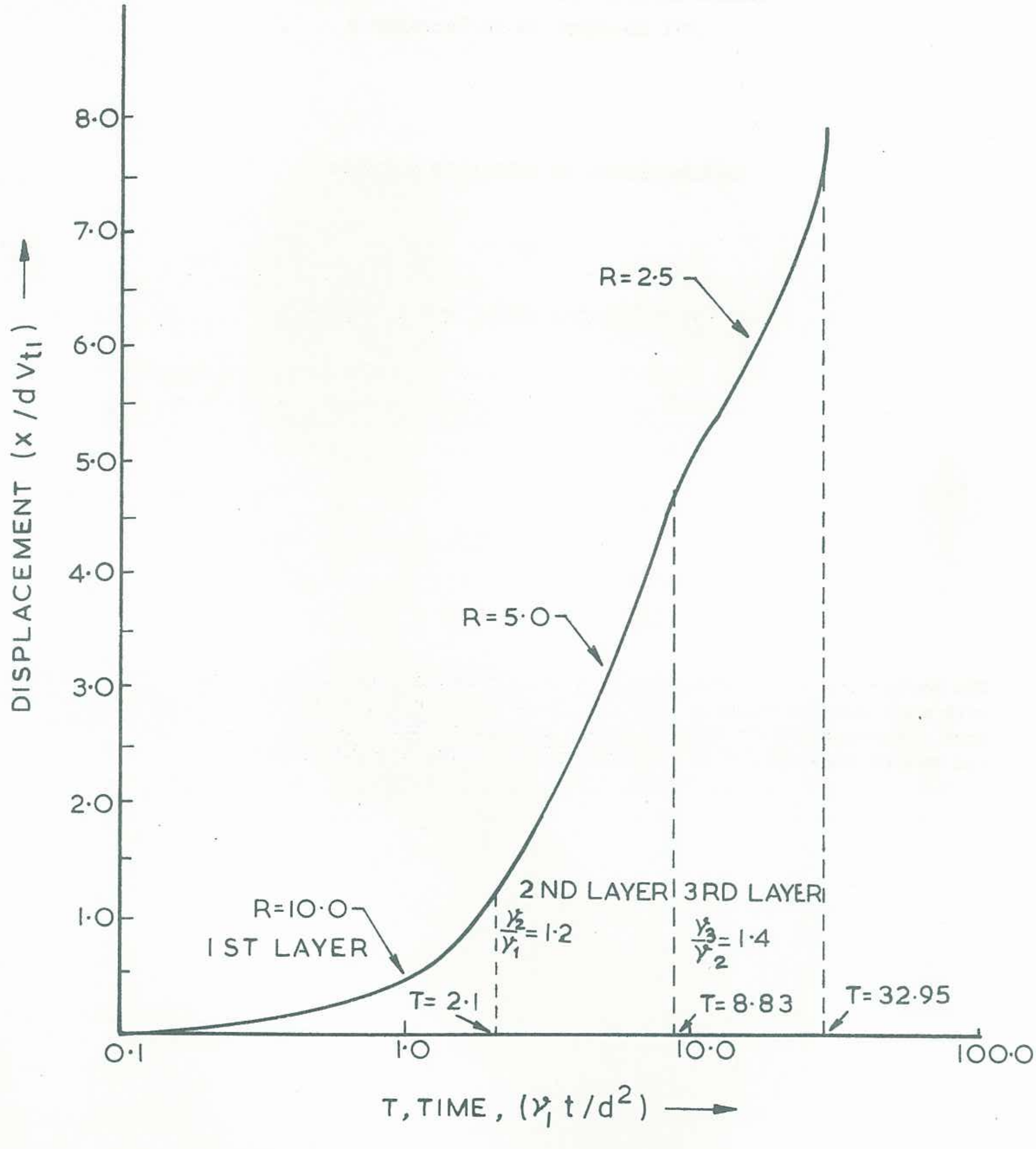

FIG. 2, DISPLACEMENT-TIME RELATIONSHIPS