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AN ENGINEERING DESIGN PROCEDURE FOR THE VELOCITY AND TEMPERATURE IN A SUPERSONIC FREE JET

by

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SUMMARY.

A semi-empirical design procedure which gives the velocity and temperature at any point in an axi-symmetrical supersonic air jet exhausting into still air is described in detail. The procedure takes into account the nozzle outlet angle and the pressure and temperature of the exhausting jet relative to the environmental conditions; thus, it applies to contoured or conical nozzles, and underexpanded or overexpanded, hot or cold, jets.

Velocities predicted by the method compare well with experimental results of the author who tested moderately underexpanded and overexpanded jets exhausting at a Mach number of 3.0 from a contoured nozzle. In these tests the plenum chamber temperature was always approximately equal to atmospheric, so that the total temperature throughout each jet was approximately equal to atmospheric. No checks have yet been made on the accuracy of the method applied to hot jets.

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NOTATION.

m_o	theoretical mass flow rate from nozzle
$m_o + m_\infty$	total jet mass flow rate through any cross-section
r	radial co-ordinate (distance from jet axis)
r_p	radius to edge of potential core
$r_{0.5}$	radius giving location of velocity 0.5 U_a at any cross-section.
x	axial co-ordinate (distance from nozzle exit)
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
F	theoretical nozzle thrust (including pressure component)
M	Mach number
P	Pressure
R	gas constant
R_o	radius of nozzle at exit
R_r	the ratio of the Reynolds number of the particular jet at nozzle exit, to that of a fully-expanded jet with total temperature equal to atmospheric (same nozzle)
T	temperature
T_t	total temperature
U	velocity
γ	ratio of specific heats (C_p/C_v)
ρ	density
σ	empirical spreading parameter, eq.(1)
θ	half-cone angle of divergent conical nozzle.

Other suffixes.

a	value on jet axis
0	value at nozzle exit
1,2,3	values on cross-sections at end of Regions I, II, III respectively
∞	value in environmental fluid.

INTRODUCTION.

The usual, more theoretical, design method for compressible turbulent jets is described briefly by Dixon et al (1). Only one experimental constant is used - a spreading parameter σ - but the solution is for fully-expanded jets only and the calculations are very long due to involved trial and error procedures. Abramovich (2) uses a basically similar approach and gives graphical solutions for fully-expanded and "off-design" jets, but the solutions for supersonic jets do not agree well with experimental results. Other work in this field includes the methods proposed by Piesik and Roberts (3), Donaldson and Gray (4) and Warren (5). However, these methods are of restricted application and generally require involved and tedious computations.

The present semi-empirical approach is an attempt to produce a simple engineering method for the prediction of velocity and temperature with wide application to fully-, over- and under-expanded hot or cold jets. In the method the velocity and temperature are assumed to equal the nozzle exit values within a potential core. The extent of the potential core, and the variation of the velocity and temperature on the axis beyond the potential core, are determined via the laws of momentum, energy and continuity. The radial velocity and total temperature profiles, and the mixing rate and spread of the jet and hence the shape of the potential core, are chosen empirically. This method is thus similar to that of Donaldson and Gray (4) except that they determine the spread of the jet following Warren (5) by taking a momentum balance across a part of a jet and using an empirical local mixing factor.

1. ASSUMED CHARACTERISTICS OF THE JET.

1.1. Potential core.

Within a cone-like core, as illustrated in figure 1, the velocity and temperature are assumed to be constant at the theoretical nozzle exit values. This has been well justified experimentally for the fully-expanded jet.

1.2. Regions.

The jet is divided into four regions along the axis as shown in figure 1.

Region I - for an under-expanded jet, from the nozzle exit to station 1 (i.e. the first half pressure cycle); for an over-expanded jet from the nozzle exit to the cross-section of minimum diameter (i.e. the unrepeated compression length).

Region II - from the end of Region I to the end of the potential core.

Region III - from the end of the potential core to the "sonic point" (i.e. the point on the axis where the velocity equals the local speed of sound).

Region IV - from the sonic point onwards.

1.3. Spreading.

The increase in the width of the mixing annulus around the potential core, and the increase in diameter of the fully mixed jet beyond the end of the core, is described by the experimental parameter σ. Its precise definition differs in the various references; here it is given by

$$\sigma = \Delta x / \Delta(r_{0.5} - r_p) \qquad (1)$$

where $r_p = 0$ in Regions III and IV. σ lies between 11 and 12 for incompressible flow, according to Pai (6). Beyond the sonic point (Region IV) the flow in the jet is taken as effectively incompressible and it is assumed

$$\sigma = 11.0 \qquad (2)$$

In Regions I, II and III, the following expression for the spreading parameter closely satisfies the author's experimental results,

$$\sigma = (11.0 + 2.80\ M_o)\ (R_r)^{0.2} \qquad (3)$$

The term in the first bracket is similar to Korst and Tripp's (7) approximate relation for fully-expanded jets, while the term in the second bracket shows that the growth of the mixing region of the jet is analogous to the growth of a turbulent boundary layer.

1.4. Velocity and Total Temperature Profiles.

(a) Regions I and II, $x < x_2$

Within the potential core ($r < r_p$) the velocity and temperature are assumed to have the theoretical nozzle exit values,

$$U = U_o \qquad T_t = (T_t)_o \qquad (4)$$

The radial variations from the potential core values to the environmental values in the mixing annulus surrounding the potential core ($r > r_p$) are given by

$$U/U_o = \exp\left[-0.6932\ (r-r_p)^2/(r_{0.5} - r_p)^2\right] \qquad (5)$$

$$\text{and}\quad T_t/(T_t)_o = T_\infty/(T_t)_o + \left[1 - T_\infty/(T_t)_o\right] U/U_o \qquad (6)$$

Equation (5) is an empirical relation suggested by Warren (5) as an adaption of equation (7) below. Equation (6) is Page and Korst's relation quoted in reference (1).

(b) Regions III and IV, $x > x_2$

The velocity and temperature radial profiles in the fully mixed regions are assumed to be given by

$$U/U_a = \exp\left[-0.6932\ (r/r_{0.5})^2\right] \qquad (7)$$

$$\text{and}\quad T_t/(T_t)_a = T_\infty/(T_t)_a + \left[1 - T_\infty/(T_t)_a\right] U/U_a \qquad (8)$$

Equation (7) is the usual empirical relation for the velocity profile in these regions.

1.5. Static pressure and density.

The static pressure in Regions I and II is not required in the present design procedure. (It is of course generally required for a full description of the jet). In Regions III and IV the static pressure is assumed to be atmospheric.

The density at any point in the jet is assumed to be given by the perfect gas law, and energy relation,

$$\rho = P/RT = P/(T_t - U^2/2C_p)R \qquad (9)$$

Since the pressure is not known in Regions I and II, the density can be evaluated in Regions III and IV only.

2. LAWS OF MOMENTUM, CONTINUITY AND ENERGY.

2.1. Conservation of Momentum.

The integrated values of the momentum, including pressure component, across any cross-section of the jet are assumed to be constant at the theoretical nozzle thrust value (F) as follows

$$F = \int_0^\infty \rho U^2 2\pi r \; dr + \int_0^\infty (P - P_\infty) \; 2\pi r \; dr \qquad (10)$$

Only in Regions III and IV, where $r_p = 0$ and P is assumed to be P_∞, can the integrals be evaluated. Thus substituting for ρ, U and T_t from equations (9), (7) and (8) respectively and taking an average value of C_p and R across the cross-section and integrating gives, for any given cross-section in Regions III and IV,

$$F = C_1 \; (A \ln X - B \ln Y) \qquad (11)$$

where

$$C_1 = \pi \gamma P_\infty (r_{0.5})^2 / 0.6932 \; (\gamma - 1)\alpha \qquad Y = (B + U_a)/B$$
$$X = A/(A - U_a) \qquad \alpha^2 = 2C_p T_\infty + \beta^2/4 \qquad (12)$$
$$A = \alpha + \beta/2 \qquad \beta = [1 - T_\infty/(T_t)_a] \, 2C_p (T_t)_a / U_a$$
$$B = \alpha - \beta/2$$

2.2. Conservation of Mass.

The net mass flow rate through any cross-section is made up of the initial mass flow rate from the nozzle (m_o) plus that drawn in from the environment between the nozzle exit and the cross-section (m_∞), therefore

$$m_o + m_\infty = \int_0^\infty \rho U \, 2\pi r \; dr \qquad (13)$$

Again for a cross-section in Region III or IV this can be integrated, and gives (using equations (12)),

$$m_o + m_\infty = C_1 \; (\ln X + \ln Y) \qquad (14)$$

2.3. Conservation of Energy.

Similarly, for the enthalpy at any cross-section

$$(m \, C_p \, T_t)_o + (m \, C_p \, T)_\infty = \int_0^\infty C_p \, T_t \, \rho \, U \, 2\pi r \; dr \qquad (15)$$

and integrating for cross-sections in Regions III and IV,

$$(m \, C_p \, T_t)_o + (m \, C_p \, T)_\infty = C_1 \; (C_2 \ln X - C_3 \ln Y) \qquad (16)$$

where equations (13) have been utilised together with

$$C_2 = (A\beta/2) + C_p T_\infty \qquad C_3 = (B\beta/2) - C_p T_\infty \qquad (17)$$

Now m_∞, from equation (14) may be substituted in equation (16) to give, using equations (13 and (17),

$$m_o \left[(C_p \, T_t)_o - (C_p T)_\infty \right] / C_1 = \left[C_2 - (C_p T)_\infty \right] \ln X - \left[C_3 + (C_p T)_\infty \right] \ln Y \qquad (18)$$

3. THE DESIGN PROCEDURE.

The object of the procedure is to evaluate the velocity and total temperature profiles given in section 1.4 above. This requires the determination of (a) the extent of each region, (b) $r_{0.5}$ and r_p in Regions I and II, and (c) $r_{0.5}$, U_a and $(T_t)_a$ in Regions III and IV.

3.1. Region I, $x < x_1$.

The velocity and temperature profiles are given by equations (4), (5) and (6), x_1 and $(r_{0.5})_1$ are obtained from figure 2. Then $r_{0.5}$ between x = o and x = x_1 is determined by the construction illustrated in figure 3. Figures 2 and 3 were derived from the characteristic solutions of Love and Grigsby (8) by assuming that $r_{0.5}$ is given by their calculated jet boundary and then approximating this boundary by a circular arc. r_p is obtained from equation (1) which in this region becomes

$$r_p = r_{0.5} - x/\sigma$$

where σ is determined from equation (3)

3.2. Region II, $x_1 < x < x_2$

The velocity and temperature profiles are again given by equations (4), (5) and (6). The value of r_p at x_2 is zero; hence the value of $r_{0.5}$ at $x = x_2$ may be calculated directly from equation (11) after substitution of $U_a = U_o$ and $(T_t)_a = (T_t)_o$. The value of x_2 itself may then be calculated from equations (19) and (3). For other cross-sections in this region $r_{0.5}$ and r_p are determined by linear interpolation between their respective values at x_1 and x_2.

3.3. Region III, $x_2 < x < x_3$.

The velocity and temperature profiles are given by equations (7) and (8). $r_{0.5}$ is given by equations (19) and (3), with $r_p = 0$, while U_a and $(T_t)_a$ are determined by trial and error from the simultaneous equations (11) and (18). The end of Region III, $x = x_3$, is the cross-section where the vélocity on the axis equals the local speed of sound, that is, where the relation between U_a and $(T_t)_a$ is

$$(U_a)^2 = \gamma R T_a = \gamma R \left[(T_t)_a - (U_a)^2/2\, C_p \right] \quad (20)$$

3.4. Region IV $x > x_3$

Here as in Region III, the velocity and temperature profiles are given by equations (7) and (8) with U_a and $(T_t)_a$ again determined from equations (11) and (18). $r_{0.5}$ is given by equation (1), which for this region becomes

$$r_{0.5} = (r_{0.5})_3 + (x - x_3)/\sigma \quad (21)$$

where σ is given by equation (2).

4. EXPERIMENTS.

The author carried out tests on a 0.800 inch exit diameter contoured nozzle designed to give an air jet of $M_o = 3.0$ exhausting into the still air environment of a very large room. The fully-expanded air jet produced by this nozzle was initially parallel, with only weak initial disturbances, and with a nearly uniform velocity profile across the exit plane. Tests were carried out with plenum chamber pressures ranging from half to twice that required for a fully-expanded jet, but the chamber temperature was always approximately atmospheric. Three separate traverses, of pitot pressure, static pressure and total temperature were carried out across several half-cross-sections in the jet, up to 150 diameters from the nozzle exit.

5. COMPARISON OF TEST RESULTS WITH PREDICTIONS.

Because the tests were on "cold" jets, the total temperature throughout each jet was approximately atmospheric, so the design solutions for the temperature could not be assessed.

Figures 4, 5 and 6 compare the predicted and measured absolute velocities at various points in a fully-expanded, under-expanded and over-expanded jet, respectively. The success of the predictions in Regions I, II and III for the three jets provides justification for the semi-empirical method of determining $r_{0.5}$ and r_p in these regions, within the range of the experiments.

Further downstream, in Region IV, the calculated velocities are generally below the measured velocities. This discrepancy may be due mainly to errors in both the calculated and measured velocities. Because the actual pressure in the jet beyond the end of the potential core was below atmospheric, the predicted velocities in Regions III and IV (based on an assumption of atmospheric pressure) are approximately 5% low. Further, because of the high turbulence in these regions the measured velocities (calculated from the pitot-static readings) may be approximately 5% high; Johannesen discusses this in reference (9).

6. CONCLUSIONS.

A straightforward design method has been presented based partly on the fluid mechanics laws and partly on experimental information. The method shows promise in predicting the velocity in highly under-expanded or over-expanded jets. Test results using hot jets, highly under-expanded jets, and with conical nozzles are needed for comparison with design solutions. For general design purposes, further data are required on the static pressure distribution in all types of jet.

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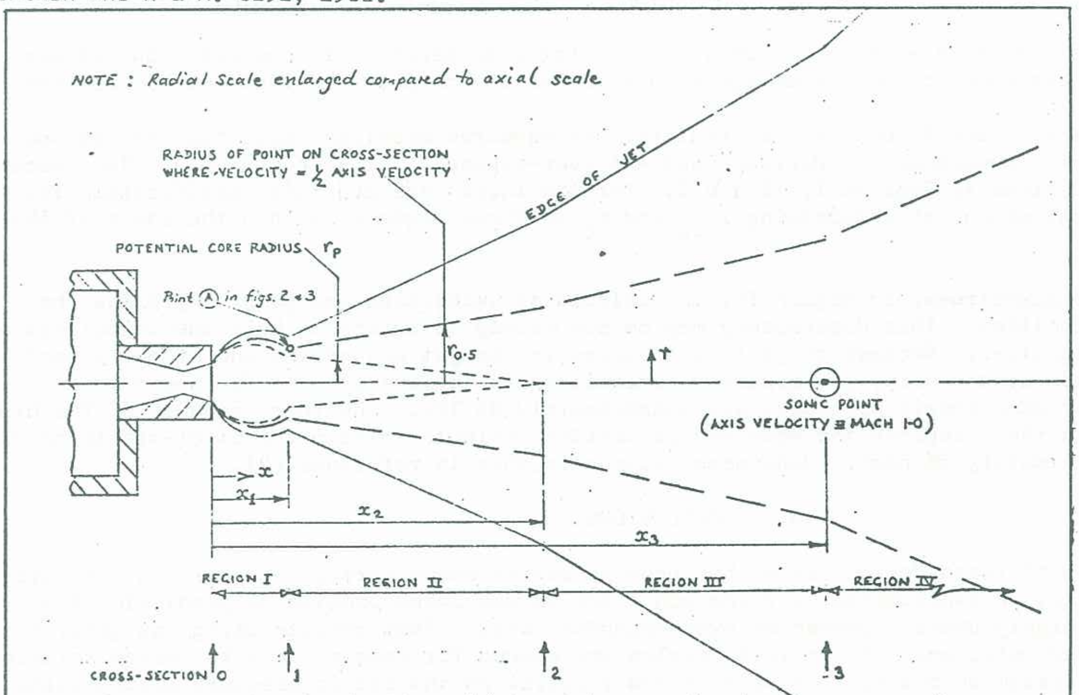

Figure 1. GEOMETRIC CHARACTERISTICS OF THE JET MODEL

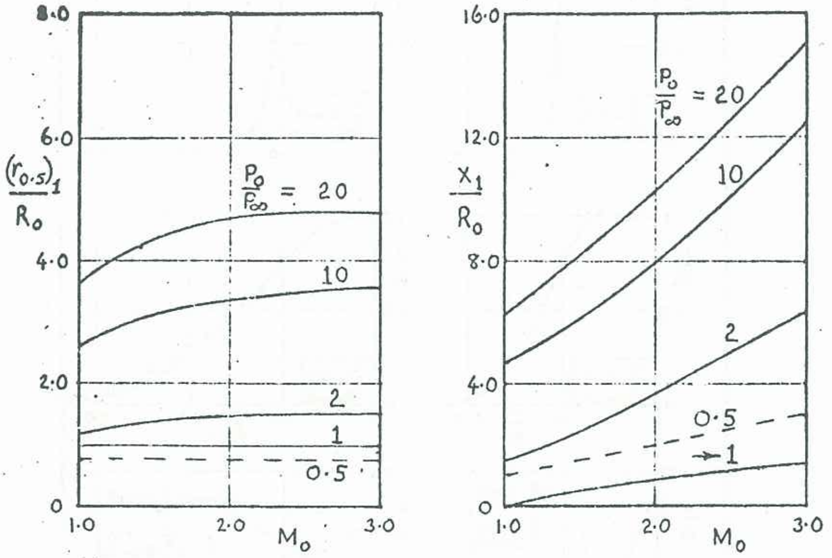

Figure 2. LOCATION OF POINT (A), [coordinates $x_1, (r_{0.5})_1$] – see fig. 3.
– from fig.17 of Reference (8) –

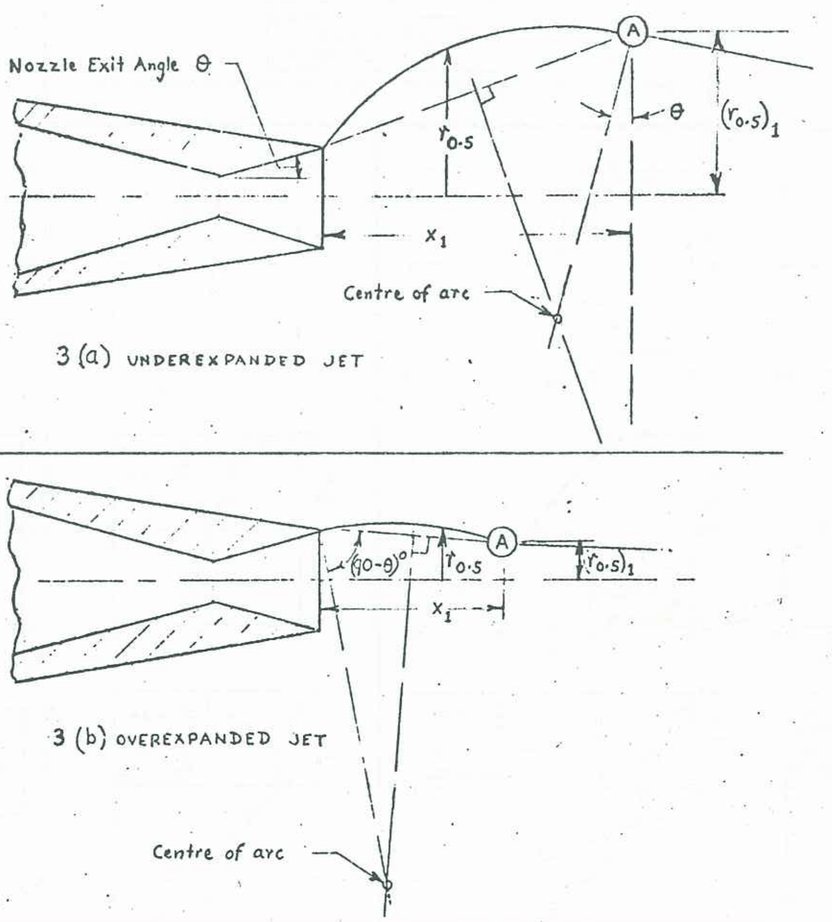

Figure 3. LOCATION OF $r_{0.5}$ IN REGION I (see fig.2 for location of A)

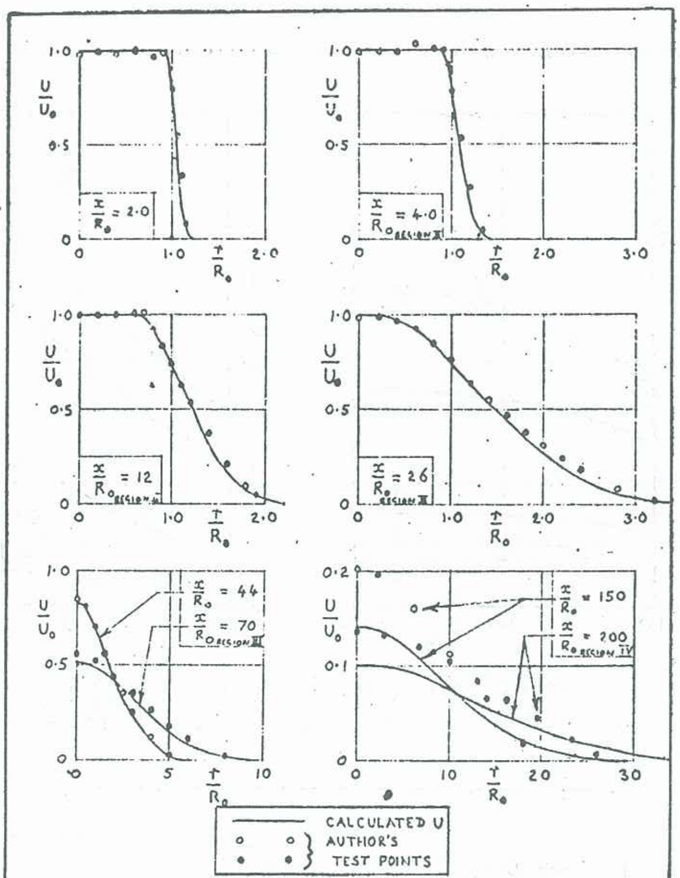

Figure 4. VELOCITY PROFILES – M_o = 3.0, FULLY-EXPANDED JET

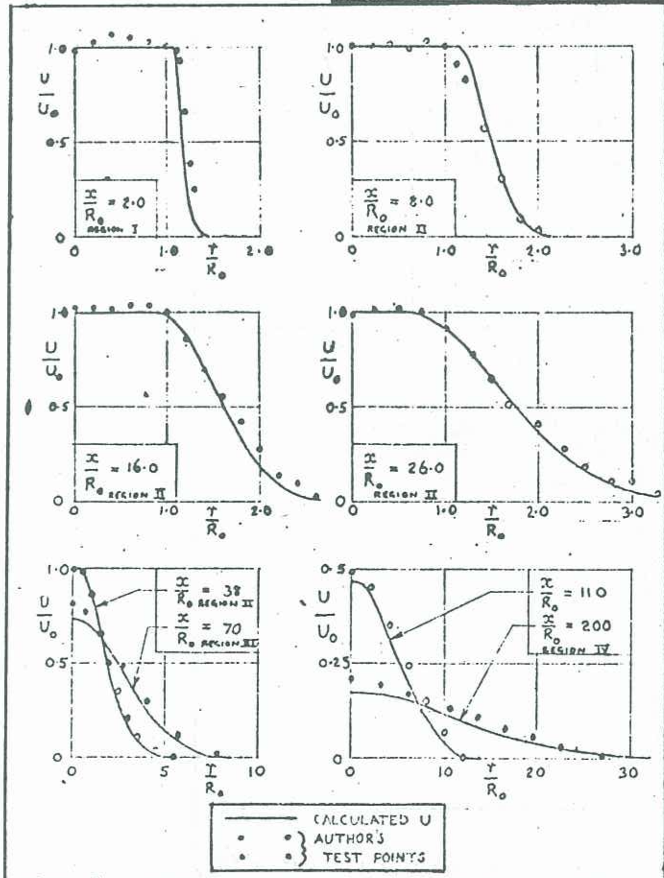

Figure 5. VELOCITY PROFILES – UNDEREXPANDED JET, M_o = 3.0 P_o/P_∞ = 1.7

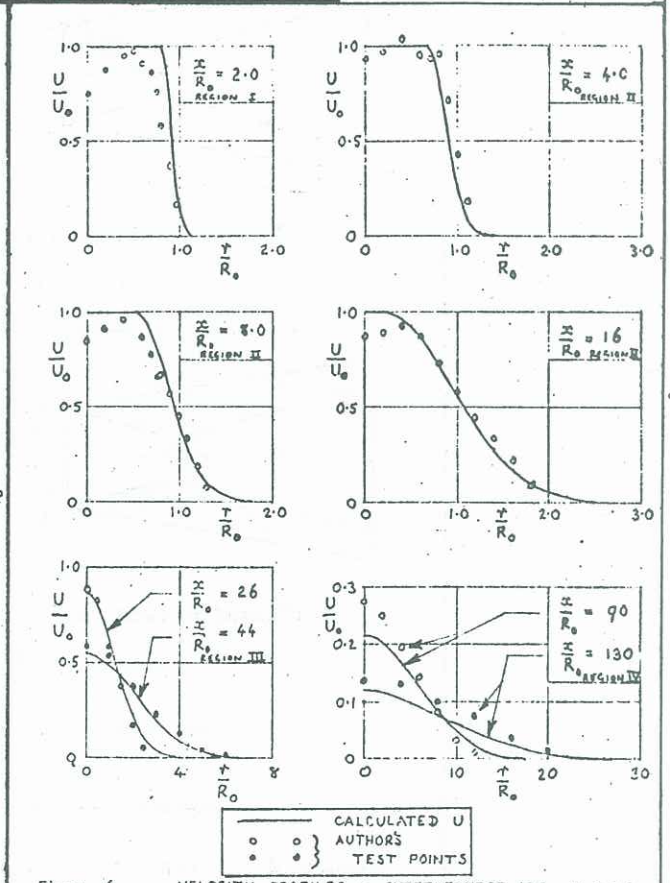

Figure 6. VELOCITY PROFILES – OVEREXPANDED JET, M_o = 3.0 P_o/P_∞ = 0.56