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Toward the Calculation and Minimization of Stokes Drag on Bodies of Arbitrary Shape

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Summary.—It would be very useful to be able to solve for slow (" Stokes ") flow of a viscous fluid past finite bodies of arbitrary shape. Somewhat less generally, we confine attention here to axially symmetric flows past bodies of revolution with arbitrary meridional sections. This problem can be reduced to solution of a pair of coupled integral equations. Although direct numerical approximation of these equations is suggested for the general case, only elongated bodies are treated here. For such slender bodies it is possible to solve analytically an approximate inverse problem and to construct families of non-trivial bodies with known drag. Some optimization problems are investigated with a view to eventual determination of the body with least drag for a given volume.

1.—INTRODUCTION

At a sufficiently low Reynolds number, steady flow of a viscous incompressible fluid is described by Stokes' equations (see, e.g., Ref. 1)

$$0 = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \qquad \text{(1.1)}$$

and the equation of continuity

$$\nabla . \mathbf{u} = 0 \qquad \text{(1.2)}$$

It may be recalled that " low Reynolds number " means physically that one or more of the following is true: (i) slow flow, (ii) small length scales, (iii) very viscous fluid. These conditions are satisfied in a great many important applications in engineering, a discussion of some of which is given in Ref. 1.

The flows in which we are interested satisfy Eqs. (1.1) and (1.2) and are uniform at infinity; i.e., where $\mathbf{i}$ is a unit vector in the x direction and U the velocity of the free stream, then

$$\mathbf{u} \to U\mathbf{i} \text{ as } \sqrt{x^2+y^2+z^2} \to \infty \qquad \text{(1.3)}$$

The velocity $\mathbf{u}$ must satisfy the no-slip condition

$$\mathbf{u} = 0 \qquad \text{(1.4)}$$

on the fixed closed surface S of a finite three-dimensional body (we exclude e.g. two-dimensional flows past cylinders, to avoid Stokes' paradox). Associated with every such finite body surface S there is a unique flow $\mathbf{u}$, pressure p, and vorticity $\omega = \nabla \times \mathbf{u}$ satisfying Eqs. (1.1) to (1.4). From this flow we can calculate in each case the vector force on the body (see Ref. 2)

$$\mathbf{F} = \int_S -p\, d\mathbf{S} + \mu\omega \times d\mathbf{S} \qquad \text{(1.5)}$$

and in particular the drag

$$D = \mathbf{i} . \mathbf{F} \qquad \text{(1.6)}$$

The drag D is thus a unique property of the geometry S, and it would be of great interest in theory and applications, to be able to calculate the manner in which D varies with S.

Unfortunately exact solutions of Stokes' equations are not numerous, even if more numerous than solutions of the full Navier-Stokes' equations at finite Reynolds number. Stokes' solution for the sphere is a good starting point, and his famous drag fomula

$$D = 6\pi \mu a U \qquad \text{(1.7)}$$

has been used in countless applications, most notably perhaps in the Millikan oil-drop experiment. Solutions for a number of other geometries are quoted by Lamb (Ref. 3) and repeated with a few additions in Ref. 1. The most important of these for our purposes is that for ellipsoids, first obtained by Oberbeck (Ref. 4). More modern and elegant treatments have since appeared (e.g., Ref. 5) for the special case of ellipsoids of revolution (spheroids); the drag formula for prolate spheroids can be written

$$D = \frac{8\pi\mu U l}{(1+\zeta^2)\operatorname{arccoth}\zeta - \zeta} \qquad \text{(1.8)}$$

where $l\zeta$, $l\sqrt{\zeta^2-1}$ are the semi-axes of the meridional ellipse.

2.—THE GENERAL AXISYMMETRIC PROBLEM

If we confine attention to bodies of revolution in axisymmetric flow, the problem can be stated in terms of a stream function $\psi(x, r)$ satisfying

$$D^4 \psi = 0 \qquad \text{(2.1)}$$

where

$$D^2 = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} \qquad \text{(2.2)}$$

The axial and radial velocity components are respectively

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad v = -\frac{1}{r}\frac{\partial\psi}{\partial x} \qquad \text{(2.3)}$$

On the body

$$r = r_0(x) \qquad \text{(2.4)}$$

the boundary conditions are

$$u = v = 0 \qquad \text{(2.5)}$$

or

$$\psi = \frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial x} = 0 \qquad \text{(2.6)}$$

while the flow at infinity is

$$\psi \to \tfrac{1}{2}Ur^2 \text{ as } \sqrt{x^2+r^2} \to \infty \qquad \text{(2.7)}$$

The boundary-value problem formulated above could be solved numerically in a number of ways. The method of finite differences (i.e., direct replacement of the differential Eq. (2.1) by a difference equation) is becoming more and more popular as computer sizes and speeds increase (see, e.g., Ref. 6) and could no doubt be used here. However, such methods are still relatively expensive in time and space on the computer, especially for problems such as the above where the mesh of points must extend to an effective infinity.

Linear boundary value problems such as the present one can be reduced to integral equations by use of appropriate singular solutions of the equation of motion, and, providing such singular solutions can be found and computed readily, this can lead to a more satisfactory numerical com-

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putation. The reduction to integral equations can be carried out formally by writing down the appropriate Green's theorem for the differential operator in a manner similar to that of Oseen (Ref. 7; see also Ref. 1, p. 79, and Ref. 2); however, more physical developments are possible.

By analogy with the corresponding problems for Laplace's equation, one might hope to represent the flow due to the body S by suitable distributions of singularities over S itself, or over some surface lying entirely within S. Again by analogy, we expect that if S itself were used there would exist a unique singularity strength for any S which generates the flow; if we use a surface lying within S there might not, for sufficiently pathological S, exist a solution to the resulting integral equation. For bodies of revolution it is particularly convenient to use distributions of singularities on the axis of revolution itself, writing (as in Ref. 8)

$$\psi = \tfrac{1}{2}Ur^2 + \tfrac{1}{2}\int_{-l}^{l} \frac{(x-\xi)\,a'_1(\xi)\,d\xi}{[(x-\xi)^2+r^2]^{\frac{1}{2}}} - \tfrac{1}{2}r^2\int_{-l}^{l}\frac{a_2(\xi)d\xi}{[(x-\xi)^2+r^2]^{\frac{1}{2}}} \quad (2.8)$$

Eq. (2.8) represents the flow as the sum of the free stream $\frac{1}{2}Ur^2$, a distribution of sources of strength proportional to $a'_1(x)$, and of " Stokeslets "* of strength proportional to $a_2(x)$, over a segment $(-l, l)$ of the axis $r = 0$ lying entirely within S. Use of the boundary conditions on the body in the form (2.6) leads to the following pair of coupled integral equations to determine the unknowns $a'_1(x)$, $a_2(x)$

$$0 = \tfrac{1}{2}Ur_0^2(x) + \tfrac{1}{2}\int_{-l}^{l}\frac{(x-\xi)\,a'_1(\xi)\,d\xi}{[(x-\xi)^2+r_0^2(x)]^{\frac{1}{2}}} - \tfrac{1}{2}r_0^2(x)\int_{-l}^{l}\frac{a_2(\xi)\,d\xi}{[(x-\xi)^2+r_0^2(x)]^{\frac{1}{2}}} \quad \text{.........(2.9)}$$

$$0 = U - \tfrac{1}{2}\int_{-l}^{l}\frac{(x-\xi)\,a'_1(\xi)\,d\xi}{[(x-\xi)^2+r_0^2(x)]^{3/2}} - \int_{-l}^{l}\frac{a_2(\xi)\,[(x-\xi)^2+\frac{1}{2}r_0^2(x)]d\xi}{[(x-\xi)^2+r_0^2(x)]^{3/2}} \quad \text{.........(2.10)}$$

It must be remarked that we do not know *a priori* whether the above integral equations possess a solution $(a'_1(x), a_2(x))$ for any given $r_0(x)$. This depends upon whether or not we can continue analytically the exact solution (which certainly exists and is unique) for external flow past the body $r = r_0(x)$ *inside* the body, all the way to the axis $r = 0$ without encountering singularities. We do not propose to study such existence questions here, but rather view the problem in a semi-inverse manner. Certainly if there does exist a solution of the system (2.9), (2.10), it must generate through the representation (2.8), an exact and unique solution for Stokes flow past the body $r = r_0(x)$.

In spite of the above warning, it would appear to be a worthwhile endeavour to solve Eqs. (2.9), (2.10) numerically, replacing the integrals by suitable quadratures and inverting a kernel matrix, as done in Ref. 2 for a time-dependent two-dimensional problem; work is proceeding on this approach. Of course a safer direct method would be to use distributions of singularities over the surface S of the body $r = r_0(x)$ itself, rather than on the axis $r = 0$, for then we should expect (although no proof is known to this writer) that a unique solution of the resulting integral equation would exist for any $r_0(x)$.

General inverse methods, in which $a'_1(x)$, $a_2(x)$ are given and $r_0(x)$ is unknown, are distinctly unpromising. For not only is the system (2.9), (2.10) non-linear when $r_0(x)$ is treated as the unknown, but more basically we must solve two equations with only one unknown. Clearly we cannot prescribe $a'_1(x)$ and $a_2(x)$ independently of each other, and would have to leave one of them, with $r_0(x)$, as an unknown. However, in an approximate version of the equations for a slender body, discussed in the following section, $a'_1(x)$ drops out and we are able to solve directly for $r_0(x)$ in terms of $a_2(x)$.

Finally we should remark that there is no need to go through the steps indicated by Eqs. (1.5), (1.6) in order to calculate the drag. We need merely make use of a formula of Payne and Pell (Ref. 5)

$$D = 4\pi\,\mu \lim_{r\to\infty} \frac{\sqrt{x^2+r^2}}{r}\left(\tfrac{1}{2}Ur^2 - \psi\right)$$

giving the drag in terms of the behaviour of the stream function at infinity to show that

$$D = 4\pi\,\mu\int_{-l}^{l} a_2(x)\,dx \quad \text{....................(2.11)}$$

Thus once we have solved for the Stokeslet distribution $a_2(x)$, the drag is obtained by simple quadrature. It is often convenient to think of a " unit " Stokeslet as having unit drag. Eq. (2.11) then simply indicates that the total drag is proportional to the total Stokeslet strength.

*The concept of a " Stokeslet " was first introduced by Hancock (see Ref. 10, p. 152); it plays a role in Stokes' flow very similar to the role of sources and sinks in potential flow. An isolated " Stokeslet " has a stream function proportional to $r^2/(x^2+r^2)^{\frac{1}{2}}$, a solution of Eq. (2.1) having non-zero vorticity.

3.—THE SLENDER-BODY APPROXIMATION

We now consider the simplifications of the representation (2.8) and integral equations (2.9), (2.10) which result from the assumption that the body $r = r_0(x)$ is slender, that is, that $r_0(x)/l$ is small. The analytical apparatus required for making this approximation was developed and discussed in Ref. 8, where it was shown that Eq. (2.8) reduces to

$$\psi = \tfrac{1}{2}Ur^2 + a_1(x) + [a_2(x) - \tfrac{1}{2}a_1''(x)]r^2\log r + [b_2(x) - \tfrac{1}{2}b_1''(x) + \tfrac{1}{4}a_1''(x)]r^2 + O(a_{1,2}\,r^4\log r) \quad \text{.........(3.1)}$$

when r is small, where

$$b_{1,2}(x) = -a_{1,2}(x)\log[2(l^2-x^2)^{\frac{1}{2}}] + \tfrac{1}{2}\int_{-l}^{l}\frac{a_{1,2}(x)-a_{1,2}(\xi)}{|x-\xi|}\,d\xi \quad \text{.........(3.2)}$$

We can proceed either by similarly approximating Eqs. (2.9), (2.10), or by applying the boundary conditions directly to Eq. (3.1), with the result that

$$a_1 = O(r_0^2), \qquad a_2 = O[(\log r_0)^{-1}] \quad \text{..................(3.3)}$$

and

$$2a_2(x)\log r_0(x) + a_2(x) + 2b_2(x) + U = 0 \quad \text{...........(3.4)}$$

Notice that a_1 does not appear in Eq. (3.4) and is nearly two orders of magnitude smaller than a_2. That is, for slender bodies the source distribution $a'_1(x)$ is of negligible importance relative to the Stokeslet distribution $a_2(x)$, and the latter may to a good approximation be chosen independently of the former in an inverse method. In Ref. 8 it was pointed out that Eq. (3.4) is a difficult singular integral equation, as a consequence of the representation (3.2) for $b_2(x)$ in terms of $a_2(x)$, and an alternative development in terms of Legendre polynomials was sought. This development, which is entirely equivalent to expanding $a_2(x)$ in a Fourier-Legendre series, leads to an infinite set of equations in an infinite number of unknowns. Unfortunately, attempts at direct inversion of a truncated set of these equations led to so-far-unexplained difficulties, and this direct approach has been temporarily abandoned.

However, Eq. (3.4) lends itself admirably to an inverse approach, for we can solve analytically for $r_0(x)$, obtaining

$$r_0(x) = \exp\left(-\frac{a_2 + 2b_2 + U}{2a_2}\right) \quad \text{..................(3.5)}$$

Now, given any $a_2(x)$ we find $b_2(x)$ by evaluating the integral transform Eq. (3.2) and substitute in Eq. (3.5) to give the corresponding body shape. Furthermore, we know the drag of such bodies by use of Eq. (2.11) and if we can construct a family of bodies by varying the Stokeslet distribution $a_2(x)$ but not the total Stokeslet strength $\int_{-l}^{l} a_2(x)\,dx$ we shall have a family with different geometric shape but constant drag.

It is convenient at this point to non-dimensionalize, writing

$$a_2(x) = UA(lx^*), \qquad r_0(x) = lR(lx^*) \quad \text{.............................(3.6)}$$

but immediately dropping the star on x^*. Then Eq. (3.5) becomes

$$R(x) = 2\sqrt{1-x^2}\,\exp\left(-\frac{A(x) + \mathcal{S}A(x) + 1}{2A(x)}\right) \quad \text{......(3.7)}$$

where

$$\mathcal{S}A(x) = \int_{-1}^{1}\frac{A(x)-A(\xi)}{|x-\xi|}\,d\xi \quad \text{.....................(3.8)}$$

Eq. (3.8) defines an " $\mathcal{S}$-transform " ($\mathcal{S}$ for " slender ") which is an integral transform of any given function $A(x)$. Recall (Ref. 8) the following property of the $\mathcal{S}$-transform

$$\mathcal{S}P_n(x) = 2\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)P_n(x) \quad \text{...........(3.9)}$$

where $P_n(x)$ is a Legendre polynomial.

For example, if $A(x) = A_0$ = constant, then $\mathcal{S}A(x) = 0$ and Eq. (3.7) gives

$$R(x) = 2\sqrt{1 - x^2}\exp\left(-\frac{1 + A_0}{2A_0}\right) \quad \text{...........(3.10)}$$

That is, this body is a spheroid with thickness/length ratio

$$\varepsilon = 2\exp\left(-\frac{1 + A_0}{2A_0}\right) \quad \text{....................(3.11)}$$

Solving for A_0, we have

$$A_0 = \frac{-1}{1 + 2\log\frac{1}{2}\varepsilon}$$

with a drag of

$$D = 8\pi\,\mu\, Ul\, A_0 \quad \text{..........................(3.12)}$$

$$= \frac{-8\pi\,\mu\, Ul}{1 + 2\log\frac{1}{2}\varepsilon} \quad \text{..........................(3.13)}$$

This result can also be obtained from the exact drag Eq. (1.8) of a spheroid by making the assumption that $\varepsilon = \sqrt{\zeta^2 - 1}/\zeta$ is small; the extent of its validity is illustrated in Fig. 1.

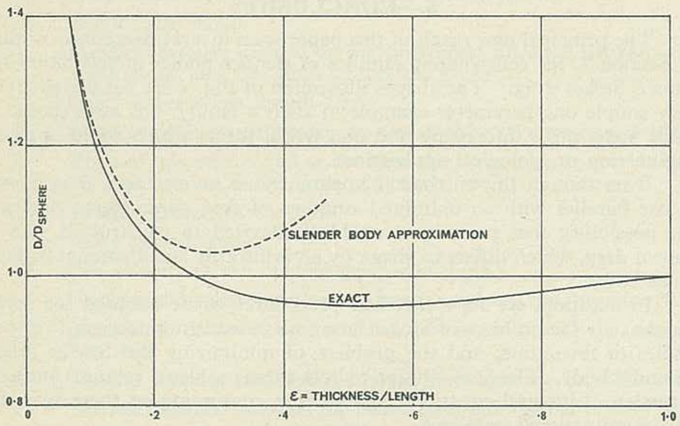

Fig. 1.—Drag of Spheroids of Varying Thickness Ratio, Scaled with Respect to the Drag of a Sphere with the Same Volume.

More generally, if

$$A(x) = \sum_{n=0}^{\infty} A_n P_n(x) \quad \text{..........................(3.14)}$$

the drag is still given by Eq. (3.12), so that for fixed A_0 and fixed length $2l$ all bodies generated by this $A(x)$ have the same drag as a spheroid of thickness/length ratio ε given by Eq. (3.11). The shape of such a family of bodies is given by

$$R(x) = 2\sqrt{1 - x^2}\exp\left[-\frac{\sum_{n=0}^{\infty} C_n P_n(x)}{\sum_{n=0}^{\infty} A_n P_n(x)}\right] \quad \text{.........(3.15)}$$

where

$$C_0 = \frac{1 + A_0}{2}$$

$$C_n = \left(\frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) A_n, \quad n = 1, 2, \ldots \quad \text{...(3.16)}$$

As a non-trivial example of such bodies consider the case

$$A_1 = A_3 = A_4 = A_5 = \ldots = 0$$

which defines the family

$$R(x) = 2\sqrt{1 - x^2}\exp\left[-\frac{\frac{1 + A_0}{2} + 2A_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right)}{A_0 + A_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right)}\right] \quad \text{...(3.17)}$$

For a finite body we must require that the denominator of the exponent be non-negative for $|x| < 1$, which requires

$$-A_0 \leqslant A_2 \leqslant 2A_0 \quad \text{.......................(3.18)}$$

The body with $A_2 = -A_0$ is cusped at its ends, while the body with $A_2 = 2A_0$ has zero thickness at its middle section.

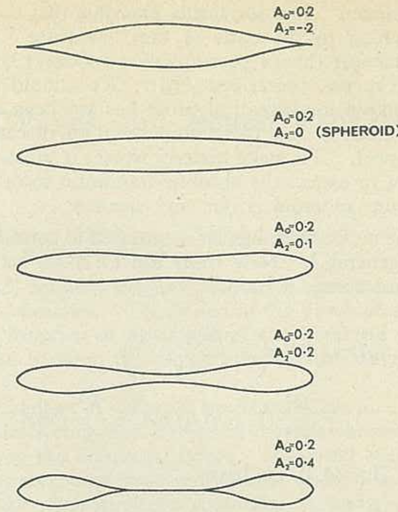

Fig. 2.—A Family of Bodies of Revolution, All with the Same Stokes Drag.

Fig. 2 gives a family of shapes calculated from Eq. (3.17) at $A_0 = 0.2$ ($\varepsilon = 0.10$), for several values of A_2. For a fixed half-length l, each of these bodies of revolution has the same drag $1.6\pi\,\mu\, Ul$, even though their geometric shape and size varies dramatically. The volume decreases as A_2 increases in this family. The cusped body ($A_2 = -0.2$) has 40% more volume than the spheroid ($A_2 = 0$), while the extreme dog-bone shape ($A_2 = 0.4$) has 43% less. Even though this particular family is presented as the simplest example only, it already includes shapes which could occur in chemical engineering or biological applications. Families with more than one parameter yield even more interesting shapes.

4.—MINIMIZATION OF DRAG

That Stokes' drag can be reduced by varying body geometry is evident from Fig. 1. Clearly at constant volume there exists a non-trivial member of the family of spheroids with least drag. This occurs at $\varepsilon = 0.52$ and gives a body with 5% less drag than the sphere of the same volume. While this can hardly be described as a large decrease, it is not insignificant and leaves open the possibility that, by allowing more freedom than just the spheroid class of geometry, we might achieve even greater reduction. It may be worth expressing this result in an alternative (perhaps more impressive!) form by observing that, for a given drag penalty, we can transport 15% more volume using a spheroid of thickness ratio 1 in 2 than by using a sphere. Nevertheless, the shallowness of the minimum is rather discouraging, and suggests that numerical methods for seeking the minimum might converge only slowly, if at all.

The mathematical problem of finding the optimum geometry under the slender-body approximation is quite easy to formulate as a problem in the calculus of variations (see, e.g., Ref. 9). The drag D is given by

$$D = 4\pi\,\mu\, Ul\int_{-1}^{1} A(x)\,dx \quad \text{........................(4.1)}$$

and the volume is

$$V = \pi\, l^3\int_{-1}^{1} R^2(x)\,dx \quad \text{..........................(4.2)}$$

We have to minimize D subject to constancy of V, with $R(x)$ and $A(x)$ related by Eq. (3.7). Suppose first of all that, in addition, the length $2l$ of the body is a constant. Then, using Lagrange multiplier λ (Ref. 9, p. 165), the condition that both D and V are stationary is

$$\delta\left(\int_{-1}^{1} R^2(x)\,dx - \lambda\int_{-1}^{1} A(x)\,dx\right) = 0 \quad \text{...........(4.3)}$$

So long as l is fixed, Eq. (4.3) holds for arbitrary λ, where the dimensionless constant λ is ultimately determined by the given value of V/l^3.

On the other hand, if we do not insist that l be fixed, we first eliminate l by minimizing $D/V^{1/3}$. The resulting variational problem is still specified by Eq. (4.3), but now λ is prescribed as

$$\lambda = 3\int_{-1}^{1} R^2(x)\,dx \Big/ \int_{-1}^{1} A(x)\,dx \quad \text{.................(4.4)}$$

At this point we should note an indication of possible trouble ahead. For if there exists a solution (an " absolute " minimum) when we do *not* fix l, then this will correspond to some particular value of the dimensionless parameter V/l^3. Suppose now we try to solve the problem with a *smaller* value of this parameter. It is physically plausible that the new solution is obtained by attaching to the ends of the "absolute" minimum body, spikes of zero thickness (hence presumably zero drag) and of length just sufficient to achieve the prescribed V/l^3. We should therefore expect (though nothing like a mathematical proof has yet been constructed) that *unless* λ is prescribed by Eq. (4.4), some kind of discontinuous solution (if any) would appear. To make matters worse, it must be observed that we have no reason to expect the absolute minimum to be a slender body; indeed the optimum spheroid is not very slender.

In spite of the above warnings let us proceed to consider the variational problem with a general λ. Now from Eq. (3.7) we can express δR^2 in terms of δA, in the form

$$\begin{aligned}\delta R^2 &= R^2\,\delta\left(-\frac{\mathcal{S}A+1}{A}\right)\\ &= \frac{R^2}{A^2}(1+\mathcal{S}A)\,\delta A - \frac{R^2}{A}\,\mathcal{S}\,\delta A \qquad \text{............(4.5)}\end{aligned}$$

Substituting into Eq. (4.3) we have

$$\begin{aligned}\int_{-1}^{1}\delta A(x)\left[\frac{R^2}{A}(1+\mathcal{S}A)-\lambda\right] &= \int_{-1}^{1}\frac{R^2}{A}\,\mathcal{S}\,\delta A\,dx\\ &= \int_{-1}^{1}\delta A(x)\,\mathcal{S}\left[\frac{R^2}{A}\right]dx \qquad (4.6)\end{aligned}$$

using the easily-proved property of the $\mathcal{S}$-transform

$$\int_{-1}^{1} F\,\mathcal{S}G\,dx = \int_{-1}^{1} G\,\mathcal{S}F\,dx$$

which holds for any functions $F(x)$, $G(x)$. Finally, from the fundamental lemma of the calculus of variations (Ref. 9, p. 185), since the variation $\delta A(x)$ is quite arbitrary, we must have

$$\frac{R^2}{A^2}(1+\mathcal{S}A)-\lambda = \mathcal{S}\left(\frac{R^2}{A}\right) \qquad \text{..............(4.7)}$$

The above result can be expressed in a remarkably simple form by defining a new function $B(x)$ via the equation

$$R^2(x) = \lambda A(x)\,B(x) \qquad \text{.......................(4.8)}$$

whereupon we have from Eq. (4.7)

$$\frac{B}{A}(1+\mathcal{S}A)-1 = \mathcal{S}B$$

or

$$A\,\mathcal{S}B - B\,\mathcal{S}A = B - A \qquad \text{....................(4.9)}$$

Even more remarkable is the result of using Eq. (3.7) to eliminate $\mathcal{S}A$ from this equation, with the result

$$1+B+\mathcal{S}B+B\log\left[\frac{\lambda AB}{4(1-x^2)}\right]=0 \qquad \text{...........(4.10)}$$

which is to be compared with

$$1+A+\mathcal{S}A+A\log\left[\frac{\lambda AB}{4(1-x^2)}\right]=0 \qquad \text{...........(4.11)}$$

obtained by re-arrangement of Eq. (3.7) itself.

One obvious consequence of the symmetry of the pair of coupled non-linear integral equations (4.10) and (4.11) is that A and B are interchangeable. Thus if there exists a solution with Stokeslet strength $A(x)$ and geometric shape defined through Eq. (4.8) by a certain function $B(x) \neq A(x)$, then there must exist another solution with a different Stokeslet strength $B(x)$ but the same shape. Unless we are prepared to admit the possibility that there is no unique Stokeslet strength for a given body shape, we must therefore conclude that any solution of the system (4.10)-(4.11) which exists will have $A \equiv B$, in which case the common function $A(x)$ satisfies the single non-linear integral equation

$$1+A+\mathcal{S}A+A\log\left[\frac{\lambda A^2}{4(1-x^2)}\right]=0 \qquad \text{...........(4.12)}$$

Note that the above is by no means a *proof* that $A \equiv B$, for it rests on the unproved assertion that the singularity distribution on the axis which generates a given body is unique. Perhaps all we should say is that we choose to seek such a restricted solution because it is intuitively more satisfactory. Clearly there are a number of quite deep mathematical questions left unanswered.

Efforts are being made to solve Eq. (4.12) numerically. This is quite a difficult task because of the logarithmic non-linearity, and no meaningful results have yet been obtained. It is possible that the difficulties mentioned after Eq. (4.3) are the root cause of the present lack of success for general (fixed) λ, in which case we should perhaps be allowing λ to vary until we obtain an absolute solution, satisfying Eq. (4.3). It may be that Eq. (4.12) defines a kind of eigenvalue problem, such that a (smooth) solution exists only for a particular value of λ.

5.—CONCLUSION

The principal new result of this paper is an inverse procedure, outlined in Section 3, for constructing families of slender bodies of revolution with known Stokes drag. The shapes illustrated in Fig. 2 are calculated from a very simple one-parameter example of such a family, but nevertheless include some quite interesting and non-trivial forms which could appear in engineering or biological applications.

Even though this method is apparently an inverse one, it is possible to use families with an unlimited number of free parameters; this raises the possibility that procedures could be devised to construct a body of known drag, which differs in shape by an arbitrarily small amount from any given body.

In addition, we have sketched procedures to be adopted for solving numerically the problem of Stokes flow past general (not necessarily slender) bodies of revolution, and the problem of minimizing the Stokes drag of a slender body. The final solution to both these problems requires numerical inversion of integral equations, and work is continuing on these very difficult computational problems.

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