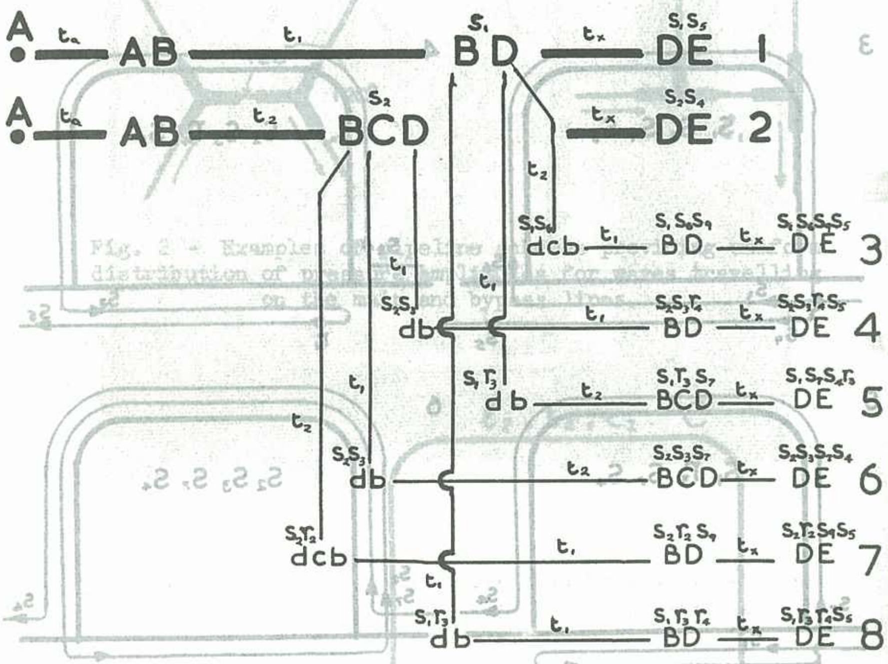

Fig. 5 - Chain representation of pressure impulses travelling through discontinuities.

PRESSURE LOSSES IN VISCOMETRIC CAPILLARY TUBES OF VARYING DIAMETER

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Kinetic energy losses of very small magnitude arise from the use of capillary tubes of slowly varying radius. The calculation of the losses in such tubes using an extension of early work by Blasius is described and applied to typical tube profiles. A tube shape giving zero kinetic energy loss is described. The range of validity of the pressure-loss equations in terms of Reynolds number is deduced from experiments on tubes of exponentially increasing radius. Comparison with available viscometric data leads to reasonable agreement.

1. Introduction

The desire to improve viscometric accuracy has recently led Caw and Wylie (1) to introduce capillary viscometers with long-flared transitions from capillary to bulb (fig. 1b). By this means the "kinetic energy" effects are made much smaller than those in standard capillaries (fig. 1a) and enhanced accuracy can

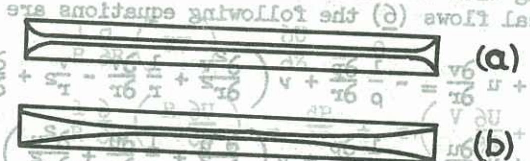

Fig. 1. Standard (a) and long-flared (b) capillaries.

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be obtained (1). To take full advantage of the new design, it would be very useful to have a theoretical estimate of the kinetic energy effects. This would, for example, enable a single-point calibration to be used over a wide range of Reynolds numbers without loss of accuracy. With the standard capillary (fig. 1a) it seems impossible to calculate the pressure losses because of the separation of the emerging fluid from the walls at the end section of the tube, but with the flared capillaries no such separation is to be expected at sufficiently low Reynolds numbers, and a calculation based on the assumption of non-separating laminar flow seems possible; at some critical Reynolds number the calculation will become invalid and separation will develop. The present paper undertakes such a calculation and presents experimental evidence to estimate the critical Reynolds number.

2. Theoretical Development

The available work on axisymmetrical flows is limited - there is nothing like the work of Fraenkel (2, 3) on which to base the present calculation because there is no simple self-similar solution in axisymmetrical flows corresponding to the Jeffrey-Hamel (4) solution. However, in view of the very small rate of divergence used by Caw and Wylie (1) it seems natural to base the calculation on the early work of Blasius (5).

The Blasius calculation (5) is not very useful as it stands, because it predicts no extra loss above the Stokes (negligible Reynolds number) loss in a symmetrical capillary starting and ending at the same diameter; this includes viscometers which have effectively infinite bulb/capillary sizes. However, by considering a further term in the Blasius development, useful results can be obtained. The authors have not found such an extension in the literature.

Starting with the Navier-Stokes equations for incompressible, axisymmetrical flows (6) the following equations are to be solved:

$$v\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad (1)$$

$$v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (2)$$

$$\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial z} = 0 \qquad (3)$$

where v, u are the velocity components in the r (radial) and z (axial) directions respectively, p is the pressure, ρ the density, and ν the kinematic viscosity.

Consider flows in a tube of slowly varying radius, in the sense that changes in radius of order R_o take place in a distance of order L, and

$$1 \gg R_o/L \qquad (4)$$

This is the Blasius assumption (5). It follows from the continuity equation (3) that if

$$u = O(\bar{u}) \qquad (5)$$

then $$v = O(\bar{u}\,R_o/L) \qquad (6)$$

where $\bar{u}$ is the mean flow velocity in the z-direction at the section where the tube radius is R_o. Using the fact, from (6), that the radial velocities are a small fraction of the axial velocities, it is found that

$$(\Delta p)_r \sim O\left(\frac{\mu\bar{u}}{L};\ \frac{\rho u^2 R_o^2}{L^2}\right) \qquad (7)$$

$$(\Delta p)_z \sim O\left(\frac{\mu\bar{u}L}{R_o^2};\ \rho\bar{u}^2\right) \qquad (8)$$

where μ is the viscosity and $(\Delta p)_r$, $(\Delta p)_z$ are typical pressure changes in the r, z directions. Thus, up to an error of order (R_o^2/L^2) p may be treated as a function of z only, and equation (1) and the final term on the right hand side of equation (2) may be ignored. Treating p(z) as the mean pressure across the section, and normalising the equations using

$$R = r/R_o;\quad Z = z/L;\quad U = u/\bar{u};\quad V = vL/R_o\bar{u};$$

$P = pR_o^2/\mu\bar{u}L$, equations (2) and (3) become

$$\frac{1}{R}\frac{\partial}{\partial R}\left(RV\right) + \frac{\partial U}{\partial Z} = 0 \qquad (9)$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right) = \frac{dP}{dZ} + \epsilon\left(V\frac{\partial U}{\partial R} + U\frac{\partial U}{\partial Z}\right) \qquad (10)$$

where $$\epsilon \equiv \bar{u}R_o^2/\nu L = Q/\pi\nu L \qquad (11)$$

and Q is the volumetric flow rate through the tube. Note that the relevant Reynolds number (ε) is based on the characteristic length of the tube; ε is considered a small parameter, and expansions in terms of ε are assumed:

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \ldots \qquad (12)$$

with similar forms for V and P. It is easily verified that no singular perturbation problems (7) arise; the viscous terms may be made to dominate the inertia terms everywhere in the flow for small ϵ.

The problem of integrating (9) and (10) using expressions like (12) is trivial; only quadratures are involved. If the dimensionless tube radius is G(Z), then the zero order ($\epsilon = 0$; Stokes flow) solution is obtained by integrating (10) after multiplying by R then repeating the process to obtain U_0. The two arbitrary functions are determined from the zero of velocity on the boundary and the zero of $\partial U_0/\partial R$ on R = 0. Multiplication by $2\pi R$ and a further integration gives the discharge; this is equal to unity in the dimensionless form chosen. Thus dP_0/dZ is found to be

$$\frac{dP_0}{dZ} = -8\,G^{-4} \tag{13}$$

This is the familiar Poiseuille law in differential form. U_0 is already determined; equation (9) finds V_0:

$$U_0 = 2\,G^{-2}\,(1 - R^2G^{-2}) \tag{14}$$

$$V_0 = 2\,G'G^{-2}R\,(1 - R^2G^{-2}) \tag{15}$$

where $G' = dG/dZ$. Inserting U_0, V_0 into the r.h.s. of equation (10) and equating terms of order ϵ yields a further equation which may be integrated in the same way. Using the fact that

$$\int_0^G U_1 R\,dR = 0$$

the Blasius (5) result is found:

$$\frac{dP_1}{dZ} = 4\,G'G^{-5} \tag{16}$$

$$U_1 = G'G^{-3}\left(\frac{2}{9} - R^2G^{-2} + R^4G^{-4} - \frac{2}{9}R^6G^{-6}\right) \tag{17}$$

$$V_1 = RG'^2G^{-4}\left(\frac{1}{3} - \frac{5}{4}R^2G^{-2} + \frac{7}{6}R^4G^{-4} - \frac{R^6G^{-6}}{4}\right) - RG''G^{-3}\left(\frac{1}{9} - \frac{R^2G^{-2}}{4} + \frac{R^4G^{-4}}{6} - \frac{R^6G^{-6}}{36}\right) \tag{18}$$

Substitution of equations of type (12) into (10) and equating powers of ϵ^2 gives the equation

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U_2}{\partial R}\right) = \frac{dP_2}{dZ} - 4G'^2G^{-6}\left(\frac{5}{9} - \frac{53}{18}R^2G^{-2} + \frac{19}{4}R^4G^{-4} - \frac{26}{9}R^6G^{-6} + \frac{19}{36}R^8G^{-8}\right) + 2G''G^{-5}\left(\frac{2}{9} - R^2G^{-2} + \frac{3}{2}R^4G^{-4} - \frac{8}{9}R^6G^{-6} + \frac{R^8G^{-8}}{6}\right) \tag{19}$$

Integration gives the result for dP/dZ as

$$\frac{dP}{dZ} = -8G^{-4}\left(1 - \frac{\epsilon G'}{2G} - \frac{121}{2160}\frac{\epsilon^2G'^2}{G^2} + \frac{11}{720}\frac{\epsilon^2G''}{G} + O(\epsilon^3)\right) \tag{20}$$

Equation (20) is correct up to terms in ϵ^3 or $(R_0/L)^2$, whichever is largest. A single integration enables the pressure loss through any shape tube of small slope and curvature to be determined; in a symmetrical tube the terms in ϵ, ϵ^3 disappear, leaving only the even powers of ϵ. Finally, the following form may be found:

$$p_B - p_A = \frac{\mu Q}{\pi R_0^4}\left[\int_A^B G^{-4}dz + \frac{\epsilon L}{8}\left(G_B^{-4} - G_A^{-4}\right) + \frac{11}{540}\epsilon^2L^2\int_A^B\left(\frac{dG}{dz}\right)^2\frac{dz}{G^6} + \frac{11}{720}\epsilon^2L^2\left(\frac{1}{G^5}\frac{dG}{dz}\Big|_B - \frac{1}{G^5}\frac{dG}{dz}\Big|_A\right)\right] \tag{21}$$

Writing (21) in terms of z and ϵL instead of Z and ϵ shows that the choice of L is arbitrary. In fact, by using a new coordinate $\zeta = Z/\epsilon$, one can find a single equation for U and $\pi (\equiv P/\epsilon)$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right) = \frac{d\pi}{d\zeta} + U\frac{\partial U}{\partial \zeta} - \left(\frac{1}{R}\int_0^R R\frac{\partial U}{\partial \zeta}dR\right)\frac{\partial U}{\partial R} \tag{22}$$

Repeated integration, using an initially parabolic profile, will generate the series solution given above. The solution is still limited to small ϵ, however, because the effect of writing $\zeta = Z/\epsilon$ is to compress the axial coordinate for high Reynolds numbers, so that the wall slopes become steeper and terms of order R_0^2/L^2 cannot be ignored.

For symmetrical tubes such that $G_A = G_B$ in (21) the first change in pressure drop is due to terms of order ϵ^2. These too will be zero if, from (20),

$$\frac{11}{3}\,\frac{G'^2}{G^2} = \frac{G''}{G} \tag{23}$$

Integration yields

$$G = (1 \pm Z/L)^{-3/8} \tag{24}$$

This shape, taking the negative sign, is shown in fig. 2. It is close to that used by Caw and Wylie (1) over part of the range of Z/L.

3. Applications

Assuming symmetry, the pressure differences in various shapes may be estimated. All cases give a pressure difference of the form

$$\Delta p = K'\mu Q\left[1 + K\epsilon^2\right] \tag{25}$$

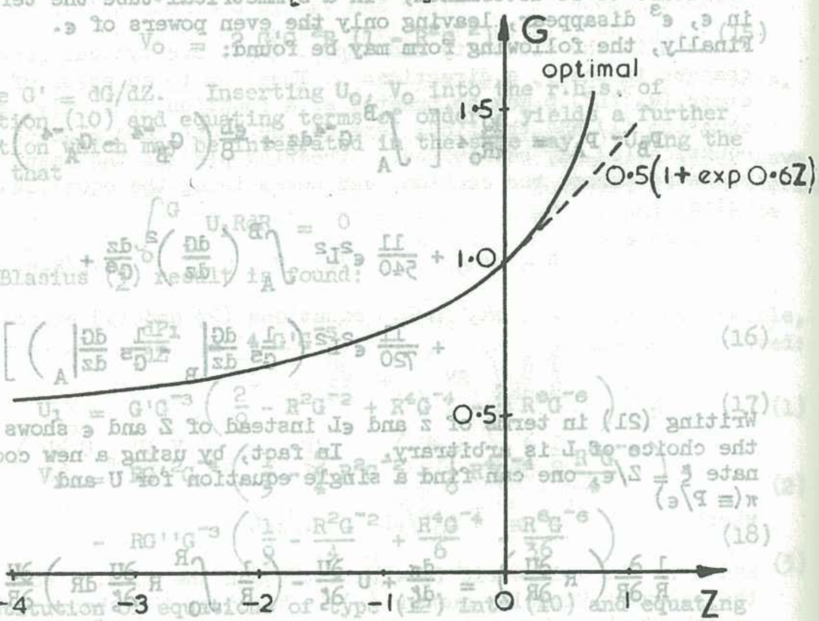

Fig. 2. Tube shape for zero kinetic energy losses.

where K, K' are constants. K is given below (Table 1) for three shapes of interest, all of which have G = 1 at Z = 0.

Shape	Range of Z	K
G = 1 + Z (cone)	0 to ∞	- .0336
G = exp. Z (8)	0 to ∞	- .0407
G = (1 + exp. Z)/2 (1)	- 3 to ∞*	+ .0003
$G = (1 - Z)^{-3/8}$ (Eqn. 24)	any	.0000

Table 1. Pressure drop factors for various shapes.

Thus K may be positive or negative, depending on the shape. For a viscometer, (25) shows that the expected relation between the viscosity and flow time will be of the form

$$\mu = At - Bt^{-3} + .. + O(t^{-7}) \tag{26}$$

where A is the usual positive calibration constant and B is given by

$$B = +K\mathcal{V}^2\rho^2/\pi^2L^2A \tag{27}$$

where $\mathcal{V}$ is the volume of fluid discharged in time t. Further detailed calculation based on equation (21) gives the theoretical curve for comparison with previous experimental results (1). Fig. 3 shows that excellent agreement between theory and

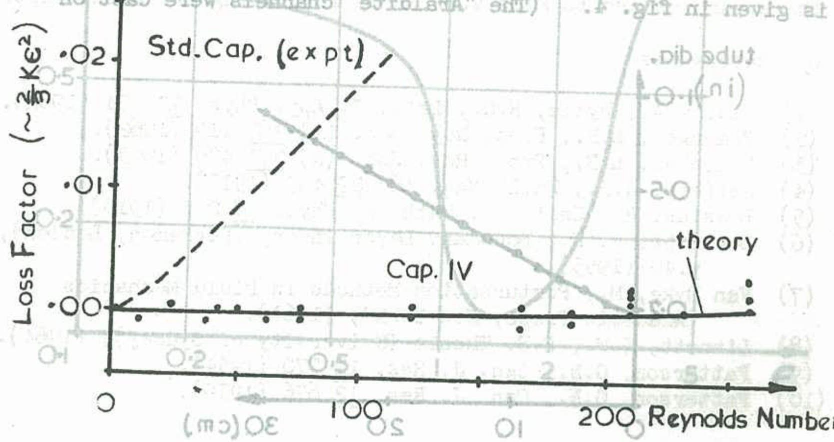

Fig. 3. Comparison between experiment and theory for capillary viscometer tube IV, reference (1).

*This range has been chosen to fit the results from capillary IV, reference (1).

experiment occurs below a Reynolds number of about 200. (In the present case the variable used as ordinate by Caw and Wylie (1) in their fig. 5 has been replaced by the close approximation $2/3\ K\ \epsilon^2$.) The large losses in standard capillaries may be noticed. There is also disagreement at large flow rates (not shown), where it was found (1) that the second term in (26) is better represented as $-Bt^{-4}$. The approximations used in analysis are invalid in this region, and no agreement can be expected. The value of ϵ corresponding to the Reynolds number (based on capillary diameter) of 200 is about 2, which is perhaps a larger value than might be expected for validity of formula (21). For example Patterson (9, 10) using two-dimensional tubes of exponentially increasing width, showed experimentally that the criterion for separated flow was approximately

$$R_s = \frac{\bar{u}w}{\nu}\frac{dw}{dz} \approx 1 \qquad (28)$$

where w(z) is the half-width of the channel. Blasius'(5) calculation gave values of R_s about four times too large, and thus prediction of R_s is unsafe.* In order to check on the value of R_s in axisymmetrical flow an experiment on a tube of (approximately) exponentially increasing radius was performed (8). The exponential shape is convenient because the value of R_s

$$\frac{\bar{u}R_0^{\,2}G}{\nu}\frac{dG}{dz} \equiv \frac{QG'}{\pi\nu GL}$$

is simply ϵ for this shape. The actual profile as measured is given in fig. 4. (The "Araldite" channels were cast on

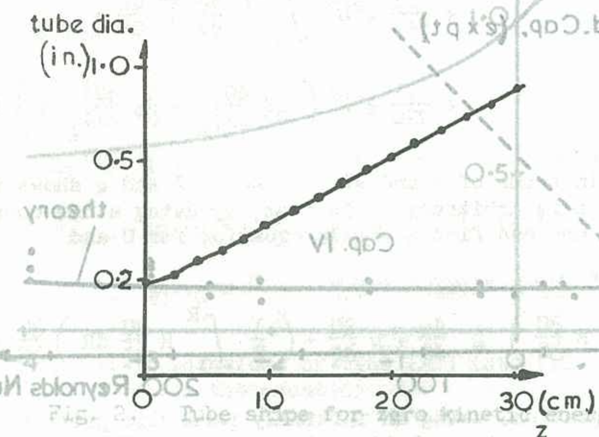

Fig. 4. Actual tube in profile.

* Actually other criteria including G'', G''', etc. are involved in determining the critical Reynolds number.

a polished steel mandrel and a parallel inlet portion was arranged before the test section.) Despite great care in manufacture and testing the results at predicting pressure losses from the measured shape were only fair; the Stokes loss and the Blasius component could be estimated within 5% or so but no useful estimate of the second-order effects could be found; the scatter on data was rather larger than the second-order effects; these in turn were smaller then expected due to slight deviations from the designed exponential profile. In fact, the extreme sensitivity of the integrals (21) to small profile changes suggests that only quite low accuracy (± 20%) can be expected in predicting second-order effects. However, this is not a great drawback when dealing with shapes showing very small second-order effects.

Of more interest is the value of ϵ at which the assumed flow pattern breaks down. Fig. 5 shows the pressure loss as a function of ϵ for the diffusing section of the capillary.

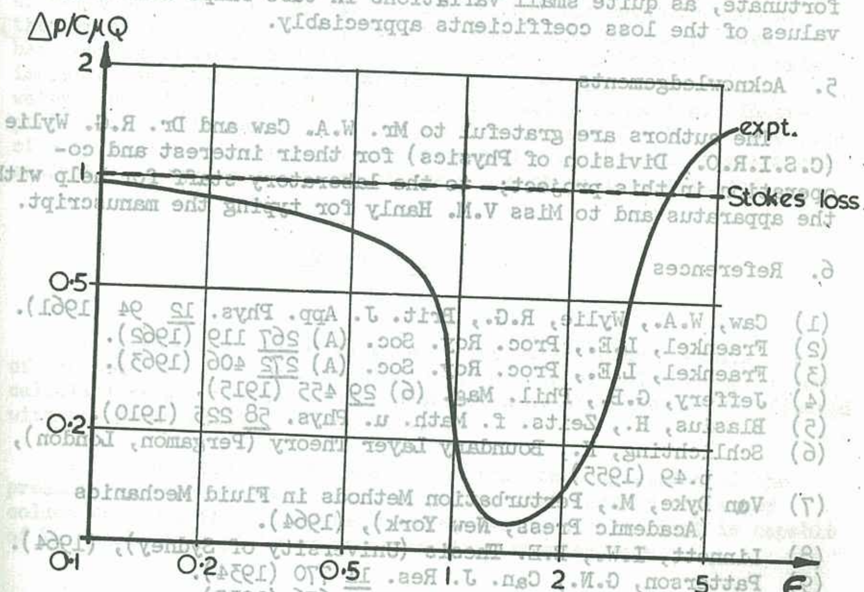

Fig. 5. Pressure-loss as a function of ϵ for exponential channel.

It is clear that $\epsilon \approx 1$ is the limit of Blasius-type flow for this particular tube. The reason for the persistence of Blasius flow in the viscometer experiment (1) may lie in the fact that for given ϵ, the value of R_s is lower in a tube in which $G(Z) = 1 + \exp. Z$ than in a tube with $G = \exp. Z$, due to the G'/G factor. Hence it appears reasonable to define

$$R_s = \epsilon\, G'/G \approx 1 \tag{29}$$

as the approximate limit of Blasius type flow and the region of validity of equation (21).

4. Conclusion

The foregoing analysis enables a useful estimate to be made of kinetic energy effects in long-flared capillary viscometers (1) up to a critical Reynolds number. This limiting Reynolds number (29) is roughly in agreement with those deduced by Patterson (9, 10) in a two-dimensional tube. Since the kinetic energy effects are extremely small up to the separation values, a fairly inexact calculation appears to be adequate for viscometric purposes. This is fortunate, as quite small variations in tube shape affect the values of the loss coefficients appreciably.

5. Acknowledgements

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RUPTURE OF THE WATER COLUMN

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ABSTRACT

Surges in pipes leading to a rupture of the water column may result in vapour cavities at regular intervals away from the control or valve end, due to a special mechanism for the establishment of the hydraulic gradient for flow. The complexity of the problem has been revealed in a study of the growth and collapse of a single isolated cavity where a rarefaction wave occurring in the normal water hammer cycle was the means for the cavity formation. Interpretations of the water column rupture phenomenon for the simple case of a horizontal pipeline will be discussed with the aid of graphical analysis and the implications of the research will be outlined.

INTRODUCTION

The phenomenon called rupture of the water column remains one of the most difficult engineering problems and as yet does not allow calculations and the subsequent protection of pipelines to be affected with any great feeling of security.

It is recognised in a general way that the depression of the pressure to vapour pressure in a pipeline will rupture the water column and this is anticipated either alongside a valve that is capable of being closed rapidly or at a knee in a pumping main.

Theoretical and experimental studies of the problem may be conveniently confined to the simplest pipe system, namely a uniform horizontal pipeline of length L terminated at the upstream and downstream ends by a reservoir and a control valve respectively.