## 5. Discussion

The agreement between theory and experiment is encouraging, but a proper test must await a controlled laboratory experiment in which the wind structure and the wave amplification are measured together.

The mechanism of amplification on a general surface displacement field is the same in essence as that described above for the simplified laboratory situation. The mean velocity and the turbulent intensities are perturbed by the water motion, each perturbation being apparent as a non-zero correlation between the water motion and the appropriate parameter of the air motion. These perturbations lead to a pressure component on the water surface in phase with the wave slope, which in turn amplifies the water wave motion.

The uncoupled action of the turbulent pressure field on the water surface is to generate waves preferentially in a single frequency band at each forward angle to the wind, (Phillips(1)). The combined effect of the two mechanisms, uncoupled and coupled, is therefore to amplify preferentially the frequency band propagating in the direction of the mean wind. This is the probable explanation for the rapid growth of a nearly-sinusoidal wave motion in the direction of the mean wind, when gusts of wind last for more than a few moments on a water surface initially at rest. The coupling with the turbulent velocity field amplifies exponentially the frequency band chosen by the turbulent pressure field.

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# ON CONVECTION IN A POROUS MEDIUM

By

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### Abstract.

Boundary layer equations are derived for convective flow of fluid over a hot plate on the bed of a semi infinite porous medium. An estimate for the heat convected away as a function of the temperature of the hot plate is used to evaluate the rate of cooling of a lava lake under a water saturated porous medium.

1. Introduction. Geothermal activity at Wairakei originates in a bed of porous water saturated volcanic debris about two and a half kilometers deep. This activity appears to be fed by a jet of hot water with a cross sectional area of two to three square kilometers and central temperatures up to two hundred and fifty degree centigrade, convected from the basement of the system.

In this paper we examine the possibility that this jet is generated on the surface of an old magma lake cooling by conduction of heat through a solid crust. We imagine water flowing in a boundary layer along the bottom and being heated as it passes over the hot crust. The bulk of the fluid above, is at a constant temperature and motionless and hence sustaining a hydrostatic pressure distribution. Since the heated fluid in the boundary layer is diminished in density the pressure on the bottom is lower than hydrostatic by a margin that increases with the thickness of this hot layer. This produces a pressure gradient along the bottom driving fluid in the direction of thickening of this layer towards a stagnation point where it leaves the lake and rises toward the surface.

We consider first a simplified problem involving steady convection in an infinite homogeneous porous medium in the half plane $z > 0$, ($z$ measures height above the plane horizontal impermeable floor of the region). The convection current is generated by a hot plate of radius $a$ in the plane of the floor at a constant temperature $T_1$ which is greater than the temperature $T_0$ of the medium sufficiently far from the plate. This problem avoids any interaction between the fluid flow over the hot plate and that induced by the jet impinging on an upper boundary or free surface.

2. **Fundamental Equations.** Take the $z$ - axis vertical with the plane $z = 0$ defining the floor of the region of fluid flow $z > 0$.

The origin of the rectangular cartesion coordinate system is at the centre of the hot plate. The symbols have the following significance.

| | |
|---|---|
| $a$ | radius of the plate |
| $e$ | porosity of the medium |
| $k$ | vertical permeability |
| $k^*$ | horizontal permeability |
| $u^*, v^*, w^*$ | components of the flow vector in the $x$-$y$-$z$-directions |
| $T(x, y, z)$ | temperature of the fluid at the point $(x, y, z)$ |
| $T_0$ | temperature of the fluid at infinity |
| $T_1$ | temperature of the plate |
| $\rho$ | fluid density at temperature $T$ |
| $\rho_0$ | fluid density at reference temperature $T_0$ |
| $\nu$ | kinematic viscosity (assumed constant) |
| $g$ | acceleration due to gravity |
| $P(x, y, z)$ | pressure in the fluid at $(x, y, z)$. |

For convenience, a modified vector $(u, v, w)$ is defined by $\rho_0(u, v, w) = \rho(u^*, v^*, w^*)$. The fluid is assumed incompressible, changing volume only as a result of change in temperature, according to the relationship;

$$\frac{\rho - \rho_0}{\rho_0} = -\alpha \frac{(T - T_0)}{(T_1 - T_0)}$$

| | |
|---|---|
| $c$ | specific heat of water |
| $c_s$ | specific heat of solid material |
| $\rho_s$ | density of solid material |

The heat capacity per unit volume of saturated porous material is written $c\rho_0 E(\rho) = (1-e)c_s\rho_s + ec\rho$.

| | |
|---|---|
| $\kappa$ | thermal conductivity of the saturated porous material |
| $\varkappa$ | diffusivity $\kappa / c\rho_0$. |



The fundamental equations governing convective flow in a saturated porous medium were derived by R.A. Wooding (1, 2).

The equation of continuity of mass conservation is

$$(2.1) \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

and the equations of motion which follow from Darcy's law are

$$\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\nu}{k^*} u = 0$$

$$(2.2) \qquad \frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\nu}{k^*} v = 0$$

$$\frac{1}{\rho_0}\frac{\partial P}{\partial z} + \frac{\nu}{k} w = -\frac{g\rho}{\rho_0}$$

Finally, the energy transport equation is

$$(2.3) \quad E\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial}{\partial x}\left(\varkappa\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varkappa\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varkappa\frac{\partial T}{\partial z}\right)$$

The time derivative in the last equation is retained at first as it leads to an estimate for the time to establish a steady convective flow over the plate. We regard the steady flow as a limiting flow approached from initial conditions, $P = P_0 - \rho_0 g z$, $T = T_0$ for $z > 0$ at $t = 0$, and boundary conditions, $w = 0$ on $z = 0$, $T = T_0$ on $z = 0$ for $r > a$, $T = T_1$ on $z = 0$ for $r < a$, where $r^2 = x^2 + y^2$.

3. **Dimensionless formulation.** Take $a$ as a unit of length and write

$$(x, y, z) \equiv a(X, Y, \beta Z); \qquad \theta \equiv (T - T_0)/(T_1 - T_0); \qquad \beta \equiv \sqrt{\frac{k}{k^*}}; \qquad \lambda \equiv \frac{\alpha g a \beta k}{\nu \varkappa}$$

$$(u, v, w) \equiv \frac{\alpha g k}{\beta \nu}(U, V, \beta W); \qquad P \equiv P_0 + \rho_0 g a \beta(\alpha h - Z); \qquad \tau = \alpha g k t / \nu a \beta E$$

Then

$$(3.1) \qquad \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0$$

$$(3.2) \qquad \frac{\partial h}{\partial X} + U = 0, \quad \frac{\partial h}{\partial Y} + V = 0, \quad \frac{\partial h}{\partial Z} + W = \theta$$

$$(3.3) \qquad \frac{\partial \theta}{\partial \tau} + U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} + W\frac{\partial \theta}{\partial Z} = \frac{1}{\lambda}\left[\beta^2\left(\frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2}\right) + \frac{\partial^2 \theta}{\partial Z^2}\right]$$

in the region $z > 0$. The boundary conditions are $h = 0$, $\theta = 0$ at $r = 0$ for $z > 0$ and, $w = 0$, $\theta = 1$ for $x^2 + y^2 < 1$, $\theta = 0$ for $x^2 + y^2 > 1$ on $z = 0$.

Consider the following estimates of the values of the physical constants relevant to the situation at Wairakei.

$$\kappa = 0.003 \text{ cms}^2/\text{sec.} \qquad k = 10^{-10} \text{ cms}^2$$

$$\alpha = 0.2 \qquad k^* = 25\,k$$

$$\nu = 1.4 \times 10^{-3} \text{ cms}^2/\text{sec.} \qquad a = 5 \times 10^5 \text{ cms.}$$

$$E = 0.5 \qquad e = 0.4$$

These give $\beta = 0.2$, $1/\lambda = 2 \times 10^{-3}$ and a time constant of 600 years corresponding to $\tau = 1$. Since the activity at Wairakei appears to have left traces of its existence dating back $10^4$ years, we assume that the lava lake has been cooled over most of this period of time by a quasi-steady convection current decaying in intensity as $T_1$ decreases with the thickening of the lava crust.

The steady state problem in cylindrical polar coordinates is

$$(3.4) \qquad \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$$

$$(3.5) \qquad \frac{\partial h}{\partial r} + u = 0, \qquad \frac{\partial h}{\partial z} + w = \theta$$

$$(3.6) \qquad u\frac{\partial \theta}{\partial r} + w\frac{\partial \theta}{\partial z} = \frac{1}{\lambda}\left[\frac{\beta^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \frac{\partial^2 \theta}{\partial z^2}\right]$$

in $z > 0$; $w = 0$, $\theta = 1$ for $r < 1$, $\theta = 0$ for $r > 1$ on $z = 0$, $h \to 0$, $\theta \to 0$ as $r \to \infty$.

These equations can be expressed in terms of a stream function $\psi$ in the usual way by writing

$$(3.7) \qquad u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

so that

$$(3.8) \qquad \begin{aligned} r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2} + r\frac{\partial \theta}{\partial r} &= 0 \\ \frac{1}{r}\left(\frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial r} - \frac{\partial \psi}{\partial r}\frac{\partial \theta}{\partial z}\right) &= \frac{1}{\lambda}\left[\frac{\beta^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \frac{\partial^2 \theta}{\partial z^2}\right] \end{aligned}$$



in $z > 0$; $\frac{\partial \psi}{\partial r} = 0$, $\theta = 1$ for $r < 1$, $\theta = 0$ for $r > 1$ on $z = 0$; $\frac{1}{r}\frac{\partial \psi}{\partial r} \to 0$, $\frac{1}{r}\frac{\partial \psi}{\partial z} \to 0$, $\theta \to 0$ as $r \to \infty$.

4. **Boundary layer equations.** If $\lambda$ is very large, we might expect the flow in most of the region $z > 0$ to be governed by the limiting system of equations

$$(4.1) \qquad \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial r} - \frac{\partial \psi}{\partial r}\frac{\partial \theta}{\partial z} = 0, \qquad r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2} + r\frac{\partial \theta}{\partial r} = 0$$

But the first equation implies $\theta = f(\psi)$, that is, $\theta$ is a function of $\psi$ and is constant along the stream lines. The second equation is elliptic of the form

$$(4.2) \qquad r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2} + r f'(\psi)\frac{\partial \psi}{\partial r} = 0$$

and satisfies the strong maximum principle (3) and hence since $\psi$ and $\theta$ are constant on a closed stream line both are constant in the region enclosed by the stream line. We conclude that the stream lines must pass through a boundary layer in which the limiting equations are not a valid approximation. If this boundary layer is confined to the hot plate on $z = 0$, the stream lines will appear to emerge from this layer as if from sources on $z = 0$. The equations (4.1) have a simple solution which is a function of $r$ only; $\frac{\partial \psi}{\partial r} + r f(\psi) = 0$. This solution appears to give the limiting behaviour of $\psi$ as $z$ tends to infinity.

The semi infinite problem at hand has no boundary condition defining a length scale in the $z$-direction or an explicit order of magnitude at any point for the stream function. Consider the transformation $z = \xi\zeta$, $\psi = \xi g$ where $\xi^3\lambda = 1$. This leaves the boundary conditions unchanged, while the differential equations become:

$$(4.3) \qquad \begin{aligned} \xi^2 r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial g}{\partial r}\right) + \frac{\partial^2 g}{\partial \zeta^2} + r\frac{\partial \theta}{\partial r} &= 0 \\ \frac{1}{r}\left(\frac{\partial g}{\partial \zeta}\frac{\partial \theta}{\partial r} - \frac{\partial g}{\partial r}\frac{\partial \theta}{\partial \zeta}\right) &= \frac{\partial^2 \theta}{\partial \zeta^2} + \xi^2\frac{\beta^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) \end{aligned}$$

If $g$ and $\theta$ are expanded as power series in $\xi$ the zero order equations are seen to be

$$(4.4) \qquad \frac{\partial^2 g_0}{\partial \zeta^2} + r\frac{\partial \theta_0}{\partial r} = 0, \qquad \frac{\partial^2 \theta_0}{\partial \zeta^2} = \frac{1}{r}\left(\frac{\partial g_0}{\partial \zeta}\frac{\partial \theta_0}{\partial r} - \frac{\partial g_0}{\partial r}\frac{\partial \theta_0}{\partial \zeta}\right).$$

In the next section we endeavour to solve these subject to the boundary conditions, $\theta_0 = 1$ for $r < 1$, $\theta \to 0$ as $r \to 1$, $\frac{\partial g_0}{\partial r} = 0$ on $\zeta = 0$, $\theta_0 \to 0$, $\frac{\partial g_0}{\partial r} \to 0$ as $\zeta \to \infty$.

5. Solution of the Boundary Layer Equation. The equations (4.4) appear to have a relatively simple approximate solution based on the supposition that the stream lines in the boundary layer are essentially parallel to the boundary $z = 0$ and so the term $\frac{1}{r}\frac{\partial g}{\partial r}\frac{\partial \theta_0}{\partial z}$ makes a secondary contribution to the equations. If this term is discarded the residual equations can be written

$$(5.1) \quad \frac{\partial^2 \theta_0}{\partial z^2} = -u\frac{\partial u}{\partial z}, \quad u = \frac{1}{r}\frac{\partial g}{\partial z}, \quad \frac{\partial^2 \theta_0}{\partial z^2} = u\frac{\partial \theta}{\partial r}$$

Thus $u^2 = -2\frac{\partial \theta_0}{\partial z}$ since $\frac{\partial \theta_0}{\partial z}$ and $u \to 0$ as $z \to \infty$, and hence

$$(5.2) \quad \frac{\partial^2 \theta_0}{\partial z^2} = \frac{\partial \theta_0}{\partial r}\Big/\sqrt{\left(-2\frac{\partial \theta_0}{\partial z}\right)}$$

This equation has a similarity solution compatible with the boundary conditions $\theta_0 = 1$ on $z = 0$ for $r < 1$, $\theta_0 \to 0$ as $r \to 1$ for $z > 0$.

Write $t = z/(1-r)^{2/3}$, $\theta_0 = 1 - \theta(t)$, and insert these variables in equation (5.2). The transformed problem is

$$\frac{d^2\theta}{dt^2} = \frac{2\sqrt{2}}{3}\, t\left(\frac{d\theta}{dt}\right)^{3/2}; \quad \theta(0) = 0, \quad \theta(\infty) = 1;$$

and has the solution

$$\theta = \frac{9}{2}\int_0^t \frac{ds}{(A + s^2)^2} \quad \text{where} \quad A = (9\pi/8)^{2/3}.$$

There seems to be no real reason for expecting this approximation to give a heat flux from the plate within fifty per cent of its correct value. On the other hand it could form a basis for a perturbation approach to the solution. This was not seriously considered as there seemed to be a more rational path to the solution expressed as a series expansion, which as a by-product, provides the solution of the convection problem for an infinite plate of width $2a$.

The basis of this second approach to the solution of equations (4.4) is the transformations and series expansions

$$(5.3) \quad r = 1 - \epsilon x, \quad z = \epsilon^{2/3} z^*, \quad \theta = \theta_0 + \epsilon\theta_1 + \cdots, \quad g = \epsilon^{1/3}(g_0 + \epsilon g_1 + \cdots)$$



by means of which we focus attention on the flow near the leading edge of the hot plate. The parameter $\epsilon$ is used here as an aid to organising the calculations and takes the value 1 in the final answer.

The zero order equations are

$$(5.4) \quad \frac{\partial^2 g_0}{\partial z^{*2}} = \frac{\partial \theta_0}{\partial x}, \quad \frac{\partial^2 \theta_0}{\partial z^{*2}} = \frac{\partial \theta_0}{\partial z^*}\frac{\partial g_0}{\partial x} - \frac{\partial \theta_0}{\partial x}\frac{\partial g_0}{\partial z^*},$$

subject to the boundary conditions, $\frac{\partial g}{\partial x} = 0$, $\theta_0 = 1$ for $x > 0$, $\theta_0 \to 0$ for $x \to 0$ on $z^* = 0$, $\frac{\partial g}{\partial x} \to 0$, $\theta_0 \to 0$ as $z \to \infty$.

These equations and boundary conditions are invariant under the similarity transformation

$$g_0 \to \delta g_0 \qquad z^* \to \delta^2 z^* \qquad x \to \delta^3 x$$

and so the solution must be a function of the invariants $g_0/\sqrt{z^*}$ and $z^*/x^{2/3}$. Let us write $s = z^*/x^{2/3}$, $\theta_0 = \theta(s)$, $g_0 = -x^{1/3}f(s)$ and substitute these new variables in equations (5.4). We find

$$(5.5) \quad \frac{d^2 f}{ds^2} = \frac{2s}{3}\frac{d\theta}{ds}, \quad \frac{d^2\theta}{ds^2} + \frac{f}{3}\frac{d\theta}{ds} = 0$$

while the boundary condition can only be reconciled with those required for $\theta_0$ and $g_0$ by demanding

$$\theta(0) = 1, \quad f(0) = 0, \quad \theta(\infty) = 0, \quad \frac{df}{ds}(\infty) = 0.$$

The prime purpose in seeking the solution of this problem is to calculate from it the heat Q convected away from the hot plate.

Now

$$Q = -2\pi a^2 K\int_0^1 r\left(\frac{\partial T}{\partial z}\right)_{z=0} dr$$

$$(5.6) \quad = 2\pi a K(T_1 - T_0)\lambda^{1/3}\left[3\frac{d\theta_0}{ds}(0) + R\right]/\beta$$

where $R$ involves $\frac{\partial \theta_n}{\partial z}(0)$, $n > 0$, and so we merely require to know $\frac{d\theta}{ds}(0)$ in order to obtain the first term of this series solution for Q.

6. Calculation of derivative $\frac{d\theta}{ds}(0)$. In order to find this derivative by the method outlined below it will be necessary to calculate simultaneously the derivative $\frac{df}{ds}(0)$.

If we define $p \equiv \frac{d\theta}{ds}$ and $q \equiv \frac{df}{ds}$, equations (5.5) can be written

$$(6.1) \qquad \frac{dq}{d\theta} = \frac{2s}{3} \qquad \frac{dp}{d\theta} = -\frac{f}{3}$$

and hence

$$(6.2) \qquad p\frac{d^2q}{d\theta^2} = \frac{2}{3} \qquad p\frac{d^2p}{d\theta^2} = -\frac{q}{3}$$

These equations are subject to the boundary conditions $p = \frac{d\theta}{ds}(0) = p_0$ say, and $q = \frac{df}{ds}(0) = q_0$ say at $\theta = 1$ and from (6.1) $\frac{dp}{d\theta} = \frac{dq}{d\theta} = 0$ at $\theta = 1$.

Now at $\theta = 0$, which corresponds to $s = \infty$ we want $p$ and $q$ to vanish. These two conditions will give us two equations from which we calculate $p_0$ and $q_0$.

For convenience we define new variables $y \equiv p/p_0$, $z \equiv q/q_0$, $\epsilon \equiv 2/3\,p_0 q_0$, $\mu \equiv -q_0^2/2p_0$, $\theta \equiv 1 - x$ and derive from equation (6.2)

$$(6.3) \qquad y\frac{d^2z}{dx^2} = \epsilon \qquad y\frac{d^2y}{dx^2} = \epsilon\mu z$$

subject to the boundary conditions $y = z = 1$, $\frac{dy}{dx} = \frac{dz}{dx} = 0$ at $x = 0$, while the equations $y = z = 0$ at $x = 1$ determine $\epsilon$ and $\mu$.

These equations are solved for $y$ and $z$ as expansions in powers of $\epsilon$ and lead to the series

$$y = 1 + \frac{\epsilon x^2}{2!}\mu + \frac{\epsilon^2 x^4}{4!}\mu(1-\mu) - \frac{\epsilon^3 x^6}{6!}\mu^2(8-7\mu) - \frac{\epsilon^4 x^8}{8!}\mu^2(127\mu^2 - 165\mu + 16) + \cdots$$

(6.4)

$$z = 1 + \frac{\epsilon x^2}{2!} - \frac{\epsilon^2 x^4}{4!}\mu + \frac{\epsilon^3 x^6}{6!}(7\mu - 1)\mu - \frac{\epsilon^4 x^8}{8!}\mu^2(127\mu - 38) + \cdots$$

Since $y$ and $z$ vanish at $x = 1$ we now have the following two equations from which to obtain $\epsilon$ and $\mu$.



$$-\frac{\delta}{2} \equiv 1 = -\frac{\epsilon}{2!}\mu - \frac{\epsilon^2}{4!}\mu(1-\mu) + \frac{\epsilon^3}{6!}\mu^2(8-7\mu) + \frac{\epsilon^4}{8!}\mu^2(127\mu^2 - 165\mu + 16) + \cdots$$

(6.5)

$$-\frac{\delta}{2} \equiv 1 = -\frac{\epsilon}{2!} + \frac{\epsilon^2}{4!}\mu - \frac{\epsilon^3}{6!}(7\mu - 1) + \frac{\epsilon^4}{8!}\mu^2(127\mu - 38) + \cdots$$

Write $\epsilon = \delta + a_2\delta^2 + a_3\delta^3 + a_4\delta^4 + \cdots$

$\mu = 1 + b_1\delta + b_2\delta^2 + b_3\delta^3 + \cdots$, insert these series in the right hand sides of equations (6.5) and equate the coefficients of powers of $\delta$. We find

$$\mu = 1 - 2\frac{\delta}{4!} + 9\frac{\delta^2}{6!} - \frac{110}{8!}\delta^3 + \cdots$$

$$= 1 + \frac{1}{6} + \frac{1}{20} + \frac{11}{12\times 42} + \cdots$$

$$\epsilon = \delta + 2\frac{\delta^2}{4!} - 7\frac{\delta^3}{6!} + \frac{254}{3.8!}\delta^4$$

$$= -2 + \frac{1}{3} + \frac{7}{90} + \frac{127}{90\times 42}$$

and with the aid of certain techniques for anticipating the sum of these series (see Shanks 4 P. 39) we obtain $\mu = 1.276..$ $\epsilon = -1.480$ and hence $p_0 = -.6300$, $q_0 = 1.0475$.

7. Estimation of the Heat Convected from the Plate. The expression (5.6) gives an approximation for the heat convected from the plate if $R$ is neglected. A corresponding expression can be derived from the equations (5.1) by transforming them according to (5.3) and evaluating the same approximate expression for the heat flux involving the first term only of the expansion. When this is carried through it is found in the case of equations (5.1) that the approximate value for $Q$ exceeds the exact value by a factor $4/3$.

It is suggested that in the case of equation (4.4), the approximation for $Q$ should be reduced by the factor $3/4$ to obtain a better estimate for the heat flux. On these grounds we take as our expression for the $Q$,

$$(7.1) \qquad Q = \frac{9\pi}{2} a K (T_1 - T_0) p_0 \left(\frac{\alpha g a}{\nu K} k^*\right)^{1/3}$$

The parameter $\alpha$ varies with temperature and we assume $\alpha = \alpha_0(T_1 - T_0)$, (for water $\alpha_0 \doteq 8 \times 10^{-4}$ per degree centigrade) so that

$$Q = Q_0 (T_1 - T_0)^{4/3} \tag{7.2}$$

8. The Cooling of a Magma Lake. We imagine that many years ago a lava lake erupted onto the bottom of the permeable trench under Wairakei and has been cooling by convection under at least a kilometer of water saturated porous material for most of the time since. Take $T_1(t)$ as the mean surface temperature of this lake at time $t$, let $a$ be its radius and $x$ the thickness of solidified crust. We will assume a linear temperature gradient in this crust and a constant temperature $T_2$ beneath, characteristic of the molten lava. Let $c_L$ be the specific heat of lava and $\rho_L$ its density.

The temperature distribution $T$ in the lava at time $t$ based on these assumptions is

$$T = T_1 - z(T_2 - T_1)/x \quad , \quad -x < z < 0, \tag{8.1}$$

where $z = 0$ is the top and $z = -x$ the bottom surface of the crust.

If $K_L$ is the thermal conductivity of the crust, the average heat lost per second from the lake is $\pi a^2 K_L (T_2 - T_1)/x$. This can be equated to the rate of change of heat in the crust, and to the rate at which heat is convected away at the surface.

$$K_L (T_2 - T_1)/x = Q_0 (T_1 - T_0)^{4/3}/\pi a^2 = \frac{d}{dt}\left[c_L \rho_L (T_2 - T_1) x/2\right] \tag{8.2}$$

In this problem $T_2$ is at least 800°C and perhaps as high as 1100°C whereas $T_1$ will be less than 400°C over most of the time of interest to us. It is therefore appropriate and convenient to substitute a constant $\Delta T$ say for $T_2 - T_1$ in equations (8.2). Write $y \equiv (T_1 - T_0)^{4/3}$, $\tau \equiv 2 Q_0^2 t/\pi^2 a^4 c_L \rho_L K_L \Delta T^2$, so that

$$\frac{d}{d\tau}\left(\frac{1}{y}\right) = y \tag{8.3}$$

and hence

$$T_1 \doteq T_0 + \left[1/2(\tau + \tau_0)\right]^{3/8}. \tag{8.4}$$



At the present time $T_1$ at Wairakei is 250°C and from this formula we get an estimate of $2 \times 10^3$ years for the time elapsed since the hypothetical eruption and a prediction of future temperatures to be expected in the jet.

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