δ_b	limit between 'mixing' zone and 'fully turbulent zone'
∇	backward difference
ϵ	eddy kinematic viscosity
ϵ_q	eddy diffusivity
η	transformed coordinate in the boundary layer
θ	dimensionless temperature in the boundary layer
μ, μ_t	molecular and eddy dynamic viscosity
ν	kinematic viscosity
ρ	density of the fluid
τ_t	turbulent shear stress
ϕ	transformed dimensionless stream function
ψ	stream function

Subscripts.

w	quantity evaluated at the wall
∞	quantity evaluated at "infinity"

Primes denote differentiation with respect to η.

THE EFFECT OF LEEWAY ON THE HYDRODYNAMICAL FORCES THAT ACT ON A BOAT

by

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SUMMARY

The paper considers the forces that act on a boat of general shape as a result of the component of its motion in the direction perpendicular to its longitudinal axis. The only restriction on the shape of the boat is that the tangent plane at any point on its surface must be nearly parallel to its vertical plane of symmetry so that it is legitimate to replace the boat by a flat plate lying in the vertical plane of symmetry of the boat, the plate and the boat having the same outline.

An approximate analysis is developed for the case when the Froude number based on the length of the boat is small. According to the zeroth order approximation the surface of the ocean acts as a reflection plate and finite wing theory may be used to calculate the forces acting on the boat.

Detailed results are given for the next approximation for a boat whose draft is either large or small compared to its length. In the former case it is found that changes in the Froude number affect the distribution of trailing vorticity over depths of the order of the length of the boat, whereas the effects of surface waves are confined to depths of the order of this length multiplied by the square of the Froude number, and have a negligible effect on the forces. In both cases it is found that the side force acting on the boat increases with the Froude number.

1. Introduction.

Considerable effort has been devoted to calculating the flow due to, and the drag force acting on a boat that moves over the surface of the ocean in the direction of its longitudinal axis. However, when the velocity of the boat has a component in the direction perpendicular to this axis the drag is altered and a side force is developed. Little attention seems to have been devoted to these effects although in some cases they may be large, as instanced by a yacht sailing across the wind.

This paper considers these effects for a boat of general shape, the only restriction being that the tangent plane at any point on the sides of the boat must be nearly parallel to the vertical plane of symmetry of the boat. Then only small errors result from replacing the boat by a flat plate lying in the vertical plane of symmetry and having the same outline as the boat.

The analysis is developed for curved as well as flat plates so that the case of a cambered strut protruding vertically from a stream is included.

2. General Theory.

Let $Ox^*y^*z^*$ be a set of rectangular axes with Oz^* vertically downwards, the origin of O being on the undisturbed surface of an ocean of infinite depth which at large distances from O is moving with uniform velocity U in the direction of Ox^*.

Suppose the surface of the ocean is pierced by a curved plate, hereafter referred to as the keel, whose displacement from the Ox^*z^* plane is everywhere small and whose projection on that plane is bounded by the curve

$$z^* = h^*(x^*) \qquad 0 \leqslant x^* \leqslant L. \tag{1}$$

Suppose that the motion due to the presence of the keel is small and that the velocity potential is $-Ux + \phi$

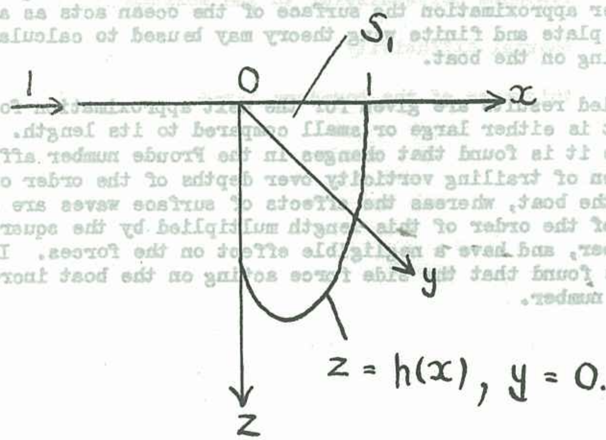


Figure 1.
Notation

Introduce the non-dimensional quantities

$$x = \frac{x^*}{L}, \quad y = \frac{y^*}{L}, \quad z = \frac{z^*}{L}, \quad \phi = \frac{\phi^*}{LU}$$

and let equation (1) in terms of these variable be

$$z = h(x) \qquad 0 \leqslant x \leqslant 1.$$

Then the acceleration potential

$$\Phi = \frac{\partial \phi}{\partial x} \tag{2}$$

must satisfy

$$\nabla^2 \Phi = 0 \tag{3}$$

throughout the region $-\infty < x < \infty$, $0 < y < \infty$, $0 < z < \infty$, and is uniquely determined throughout that region by the following conditions:

$$\frac{\partial \Phi}{\partial z} - F^2 \frac{\partial^2 \Phi}{\partial x^2} = 0 \quad \text{for } z = 0 \tag{4}$$

where $F = \frac{U}{\sqrt{Lg}}$ is the Froude number and g is the acceleration due to gravity,

$$\left.\begin{aligned} \Phi &= f(x,z) && \text{for } y = 0 \text{ and } x,z \text{ in } S_1 \\ &= 0 && \text{for } y = 0 \text{ and } x,z \text{ not in } S_1 \end{aligned}\right\} \tag{5}$$

it being supposed for the present that $f(x,z)$ is a known function,

$$\Phi \to 0 \quad \text{at large distances from } O, \tag{6}$$

the Kutta-Joukowski condition: $\Phi = 0$ (7)

at the trailing edge of the plate, and finally that there be no upstream inclined waves.

An expression for Φ may be derived very simply from equation (5) of a paper by Michell (1). This equation gives the perturbation velocity potential ϕ due to a thin ship, that is symmetrical about the Oxz plane, in terms of the values of $\frac{\partial \phi}{\partial y} = g(x,z)$ there, and is in the present notation

$$\phi = -\frac{2}{\pi^2}\int_0^\infty\int_0^\infty\int_0^1\int_0^{h(x_1)} \frac{e^{-\sqrt{n^2+w^2}\,y}}{\sqrt{n^2+w^2}} \cos nz \cos nz_1 \cos w(x-x_1) g(x_1,z_1)\,dz_1\,dx_1\,dw\,dn$$

$$+ \frac{2F^2}{\pi^2}\int_0^\infty\int_0^\infty\int_0^1\int_0^{h(x_1)} \frac{e^{-\sqrt{n^2+w^2}\,y}}{\sqrt{n^2+w^2}\,(F^4w^4+n^2)} \sin n(z+z_1) \cos w(x-x_1) g(x_1,z_1)\,dz_1\,dx_1\,dw\,dn$$

$$+ \frac{2F^4}{\pi^2}\int_0^\infty\int_0^\infty\int_0^1\int_0^{h(x_1)} \frac{e^{-\sqrt{n^2+w^2}\,y}}{\sqrt{n^2+w^2}\,(F^4w^4+n^2)} \cos n(z+z_1) \cos w(x-x_1) g(x_1,z_1)\,dz_1\,dx_1\,dw\,dn$$

$$- \frac{2F^2}{\pi}\int_{\frac{1}{F^2}}^\infty\int_0^1\int_0^{h(x_1)} \frac{w^2}{\sqrt{F^4w^2-1}}\, e^{-F^2w^2(z+z_1)} \sin[w(x-x_1)+\sqrt{F^4w^2-1}\,y]\, g(x_1,z_1)\,dz_1\,dx_1\,dw$$

$$- \frac{2F^2}{\pi}\int_0^{\frac{1}{F^2}}\int_0^1\int_0^{h(x_1)} \frac{w}{\sqrt{1-F^4w^2}}\, e^{-F^2w^2(z+z_1)-w\sqrt{1-F^4w^2}\,y} \cos w(x-x_1)\, g(x_1,z_1)\,dz_1\,dx_1\,dw .$$

Differentiating this equation with respect to y gives the values of $\frac{\partial\phi}{\partial y}$ in terms of its values, $g(x,z)$, on S_1. It is concluded that for the problem considered herein

$$\Phi(x,y,z) = I_1 + I_2 + I_3 + I_4 + I_5$$

where

$$\left.\begin{aligned}
I_1 &= \frac{2}{\pi^2}\int_0^\infty\int_0^\infty\int_0^1\int_0^{h(x_1)} e^{-\sqrt{n^2+w^2}\,y} \cos nz \cos nz_1 \cos w(x-x_1) f(x_1,z_1)\,dz_1\,dx_1\,dw\,dn \\
I_2 &= -\frac{2F^2}{\pi^2}\int_0^\infty\int_0^\infty\int_0^1\int_0^{h(x_1)} e^{-\sqrt{n^2+w^2}\,y} \frac{w^2 n}{F^4w^4+n^2} \sin n(z+z_1) \cos w(x-x_1) f(x_1,z_1)\,dz_1\,dx_1\,dw\,dn \\
I_3 &= -\frac{2F^4}{\pi^2}\int_0^\infty\int_0^\infty\int_0^1\int_0^{h(x_1)} e^{-\sqrt{n^2+w^2}\,y} \frac{w^4}{F^4w^4+n^2} \cos n(z+z_1) \cos w(x-x_1) f(x_1,z_1)\,dz_1\,dx_1\,dw\,dn \\
I_4 &= \frac{2F^2}{\pi}\int_{\frac{1}{F^2}}^\infty\int_0^1\int_0^{h(x_1)} w^2 e^{-F^2w^2(z+z_1)} \cos[w(x-x_1)+\sqrt{F^4w^2-1}\,y]\, f(x_1,z_1)\,dz_1\,dx_1\,dw \\
&\text{and} \\
I_5 &= \frac{2F^2}{\pi}\int_0^{\frac{1}{F^2}}\int_0^1\int_0^{h(x_1)} w^2 e^{-F^2w^2(z+z_1)-w\sqrt{1-F^4w^2}\,y} \cos w(x-x_1) f(x_1,z_1)\,dz_1\,dx_1\,dw .
\end{aligned}\right\} \quad (8)$$

The integrations with respect to n and w in the expression for I_1 may be resolved for

$$\left.\begin{aligned}
&\cos nz \cos nz_1 = \tfrac{1}{2}\{\cos n(z+z_1) + \cos n(z-z_1)\} , \\
&\int_0^\infty e^{-y\sqrt{n^2+w^2}} \cos n(z+z_1)\,dn = \frac{wy}{[y^2+(z+z_1)^2]^{\frac{1}{2}}} K_1[w\{y^2+(z+z_1)^2\}^{\frac{1}{2}}] , \\
&\text{and} \\
&\int_0^\infty w\, K_1[w\{y^2+(z+z_1)^2\}^{\frac{1}{2}}] \cos wx\,dw = \frac{\pi\{y^2+(z+z_1)^2\}^{\frac{1}{2}}}{2\{x^2+y^2+(z+z_1)^2\}^{3/2}}
\end{aligned}\right\} \quad (9)$$

the latter two results being given by Bateman (2).

Thus

$$I_1 = \frac{1}{2\pi}\int_0^1\int_0^{h(x_1)} \left\{ \frac{y}{[(x-x_1)^2+y^2+(z+z_1)^2]^{3/2}} + \frac{y}{[(x-x_1)^2+y^2+(z-z_1)^2]^{3/2}} \right\} f(x_1,z_1)\,dz_1\,dx \quad (10)$$

3. The case when F is small.

3.1 Zeroth order approximation.

Putting $F = 0$ in equation (8) gives $\Phi(x,y,z) = I_1$ and I_1 is given by equation (10). This shows that $\Phi(x,y,z)$ is the potential due to a double layer distributed over S_1 and $\overline{S_1}$, the image of S_1 with respect to the Oxy plane. The strength of the layer is an even function of z_1 and is $2f(x_1,z_1)$ for (x_1,z_1) in S_1. This is the familiar result of finite wing theory which thus gives the zeroth order approximation to the flow.

It is noted that when $F = 0$, (4) becomes

$$\frac{\partial\Phi}{\partial z} = 0 \qquad \text{for} \quad z = 0 ,$$

which is in agreement with the above result.

3.2 The first order approximation in F^2.

Formal approximations to the various terms in equation (8) are made and the resulting velocity potential is investigated to see if it satisfies the conditions required of it.

Thus formally,

$$I_2 = -\frac{2F^2}{\pi^2}\int_0^1\int_0^{h(x_1)}\int_0^\infty\int_0^\infty \frac{w^2}{n}\, e^{-\sqrt{n^2+w^2}\,y} \sin n(z+z_1)\cos w(x-x_1) f(x_1,z_1)\,dz_1dx_1dw\,dn$$

$$+\ O(F^4)$$

$$= \frac{2F^2}{\pi^2}\frac{\partial^2}{\partial x^2}\int_0^z\int_0^\infty\int_0^\infty\int_0^1\int_0^{h(x_1)} e^{-\sqrt{n^2+w^2}\,y}\cos n(\zeta+z_1)\cos w(x-x_1) f(x_1,z_1)\,dz_1dx_1dw\,dn\,d\zeta$$

$$+\ O(F^4)$$

$$= \frac{F^2}{\pi}\frac{\partial^2}{\partial x^2}\int^z\int_0^1\int_0^{h(x_1)} \frac{y\, f(x_1,z_1)}{[(x-x_1)^2+y^2+(\zeta+z_1)^2]^{3/2}}\,dz_1dx_1d\zeta\ +\ O(F^4) \qquad (11)$$

using the results (9),

$$I_3 = O(F^4), \qquad (12)$$

$$\text{and}\quad I_4 + I_5 = \frac{2F^2}{\pi}\,(\text{R.P.})\int_0^1\int_0^{h(x_1)} J\, f(x_1,z_1)\,dz_1dx_1 \qquad (13)$$

$$\text{where}\quad J = \int_0^\infty w^2\, e^{-F^2w^2(z+z_1)+i\,w(x-x_1+\sqrt{F^4w^2-1}\,y)}\,dw\ . \qquad (14)$$

Putting $w = \frac{w'}{F^2}$ and evaluating (14) formally by the method of steepest descent it is found that equation (13) becomes

$$I_4 + I_5 \simeq \frac{4F^2y}{\pi}\int_0^1\int_0^{h(x_1)} \frac{y^2-3(x-x_1)^2}{[(x-x_1)^2+y^2]^3}\, f(x_1,z_1)\,dz_1dx_1\ . \qquad (15)$$

This approximation to $I_4 + I_5$ is independent of z so that its contribution to Φ can be considered as being absorbed into the arbitrary lower limit of integration with respect to ζ in equation (11).

Thus the formal approximation to equation (8) given by equations (10) to (15) is

$$\Phi(x,y,z) = \frac{\partial\phi}{\partial x} = \frac{1}{2\pi}\int_0^1\int_0^{h(x_1)}\left\{\frac{y}{[(x-x_1)^2+y^2+(z+z_1)^2]^{3/2}} + \frac{y}{[(x-x_1)^2+y^2+(z-z_1)^2]^{3/2}}\right\}$$

$$f(x_1,z_1)\,dz_1dx_1$$

$$+\ \frac{F^2}{\pi}\frac{\partial^2}{\partial x^2}\int^z\int_0^1\int_0^{h(x_1)} \frac{y\, f(x_1,z_1)}{[(x-x_1)^2+y^2+(\zeta+z_1)^2]^{3/2}}\,dz_1dx_1d\zeta\ . \qquad (16)$$

the lower limit of integration for ζ in the 2nd term on the R.H.S. being taken as ∞ so that $\frac{\partial\phi}{\partial x} \to 0$ as $z \to \infty$. Integrating with respect to x from $-\infty$ to x gives

$$\phi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^x\int_0^1\int_0^{h(x_1)}\left\{\frac{y}{[(\xi-x_1)^2+y^2+(z+z_1)^2]^{3/2}} + \frac{y}{[(\xi-x_1)^2+y^2+(z-z_1)^2]^{3/2}}\right\}$$

$$f(x_1,z_1)\,dz_1dx_1d\xi$$

$$+\ \frac{F^2}{\pi}\frac{\partial}{\partial x}\int_\infty^z\int_0^1\int_0^{h(x_1)} \frac{y\, f(x_1,z_1)}{[(x-x_1)^2+y^2+(\zeta+z_1)^2]^{3/2}}\,dz_1dx_1d\zeta\ . \qquad (17)$$

Now if $f(x_1,z_1)$ in (17) is chosen to vanish at the trailing edge so that condition (7) is satisfied, $\Phi = \frac{\partial\phi}{\partial x}$ with ϕ given by (17) will satisfy each of the conditions (2) to (7) to order F^2. For example condition (4) is satisfied because to order F^2,

$$\left[\frac{\partial\Phi}{\partial z} - F^2\frac{\partial^2\Phi}{\partial x^2}\right]_{z=0}$$

is, since the first term on the R.H.S. of (16) satisfies $\frac{\partial\Phi}{\partial z} = 0$ for $z = 0$

$$\underset{z\to 0}{\text{limit}}\left\{-\frac{F^2}{2\pi}\frac{\partial}{\partial x}\int_0^1\int_0^{h(x_1)}\left\{\frac{y}{[(x-x_1)^2+y^2+(z+z_1)^2]^{3/2}} + \frac{y}{[(x-x_1)^2+y^2+(z-z_1)^2]^{3/2}}\right\}\right.$$

$$f(x_1,z_1)\,dz_1dx_1$$

$$\left.+\ \frac{F^2}{\pi}\frac{\partial}{\partial x}\int_0^1\int_0^{h(x_1)} \frac{y\, f(x_1,z_1)}{[(x-x_1)^2+y^2+(z+z_1)^2]^{3/2}}\,dz_1dx_1\right\}$$

$$= 0\ .$$

The only unacceptable feature of the expression (17) for ϕ is that the 2nd term on the R.H.S., ϕ^* say, where

$$\phi^* = \frac{F^2}{\pi}\frac{\partial}{\partial x}\int_\infty^z\int_0^1\int_0^{h(x_1)} \frac{y\, f(x_1,z_1)}{[(x-x_1)^2+y^2+(\zeta+z_1)^2]^{3/2}}\,dz_1dx_1d\zeta$$

has too strong a singularity at the origin, O.†

It may be shown that near O

$$\phi \sim \frac{F^2}{\pi}\frac{\partial}{\partial x}\int_0^1 \frac{y\, f(x_1,0)}{z+\sqrt{z^2+(x-x_1)^2+y^2}}\, dx_1$$

and that $f(x_1,0) \sim \frac{\text{const.}}{\sqrt{x_1}}$ for x_1 small, so that

$$\phi^* \sim \frac{\text{const.}\left\{P_{-\frac{1}{2}}\left(\frac{y}{(x^2+z^2)^{\frac{1}{2}}}\right) - \frac{2}{\pi}\, Q_{-\frac{1}{2}}\left(\frac{y}{(x^2+z^2)^{\frac{1}{2}}}\right)\right\}}{(x^2+y^2+z^2)^{\frac{1}{2}}} \qquad (18)$$

where $P_{-\frac{1}{2}}$ and $Q_{-\frac{1}{2}}$ are Legendre functions of order $-\frac{1}{2}$.

It is noted that (18) implies that the fluid speed $\sim \frac{1}{r^{3/2}}$ for r small where $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ so that the kinetic energy of the fluid is not bounded near O. The singularity of ϕ at O is thus too strong and must be eliminated.

This may be done by subtracting from the expression (17) a term ϕ^{**} which has the same behaviour near O as does ϕ^*, has appropriate behaviour at large distances from O and satisfies the condition (4) for $z = 0$.

However, for the present investigation ϕ^{**} may be neglected as it is easy to see that it will not give rise to a term of order F^2 in the force that acts on the boat. Firstly (17) shows that the velocity components corresponding to ϕ^{**} are at most of order F^2. Also by (4) the length scale of the motion corresponding to ϕ^{**} is of order F^2 so that the velocity components will be negligible except for depths that are less than order F^2. {The results given by Peters (3) for the (stronger) singularity corresponding to a moving pressure point show this behaviour explicitly.} It is concluded that the contribution from ϕ^{**} to the force is of order F^4 so that as stated above ϕ^{**} may be neglected. However, it is of interest to note that ϕ^{**}, but not ϕ as given by (17), represents a motion having wave like characteristics.

3.3 The Direct Problem.

Consider now the direct problem, i.e. the problem of determining the forces when the displacement $y = d(x,z)$ of the keel from the Oxz plane is given and not $f(x,z)$. Instead of the first of conditions (5) ϕ must satisfy

† This was pointed out to the author by Professor J.J. Mahony.

$$\frac{\partial\phi}{\partial y} = -\frac{\partial\, d(x,y)}{\partial x} \quad \text{for } y = 0 \text{ and } x,z \text{ in } S_1. \qquad (19)$$

Let $f(x,z) = f_o(x,z) + F^2 f_1(x,z) + o(F^2)$ (20)

and substitute (17) with f given by (20) into (19). Terms of zero order in F give

$$-\frac{\partial\, d(x,z)}{\partial x} = \frac{1}{2\pi}\lim_{y\to 0}\frac{\partial}{\partial y}\int_{-\infty}^{x}\int_0^1\int_0^{h(x_1)}\left\{\frac{y}{[(\xi-x_1)^2+y^2+(z+z_1)^2]^{3/2}} + \frac{y}{[(\xi-x_1)^2+y^2+(z-z_1)^2]^{3/2}}\right\}$$

$$f_o(x_1,z_1)\, dz_1 dx_1 d\xi, \quad \text{for } x,z \text{ in } S_1. \qquad (21)$$

Terms of order F^2 give

$$-\frac{\partial\, d_1(x,z)}{\partial x} = \frac{1}{2\pi}\lim_{y\to 0}\frac{\partial}{\partial y}\int_{-\infty}^{z}\int_0^1\int_0^{h(x_1)}\left\{\frac{y}{[(\xi-x_1)^2+y^2+(z+z_1)^2]^{3/2}} + \frac{y}{[(\xi-x_1)^2+y^2+(z-z_1)^2]^{3/2}}\right\}$$

$$f_1(x_1,z_1)\, dz_1 dx_1 d\xi, \quad \text{for } x,z \text{ in } S_1 \qquad (22)$$

where $$\frac{\partial\, d_1(x,z)}{\partial x} = \frac{1}{\pi}\lim_{y\to 0}\frac{\partial^2}{\partial x\partial y}\int_{\infty}^{z}\int_0^1\int_0^{h(x_1)}\frac{y\, f_o(x_1,z_1)\, dz_1 dx_1 d\zeta}{[(x-x_1)^2+y^2+(\zeta+z_1)^2]^{3/2}}. \qquad (2$$

Each of equations (21) and (22) is a particular case of the integral equation o lifting surface theory. Equation (21) is used to determine f_o which is substituted into equation (23) to give $\frac{\partial d_1}{\partial x}$. Finally equation (22) is used to determine f_1.

Equations (21) to (23) may be considerably simplified if the draft of the boat is either large or small compared to its length, and these cases are considered in turn.

3.3.1 Case of boat whose draft is large compared to its length.

Consider the case when S_1 is the rectangular region $0 < x < 1$, $0 < z < H$ where $H >> 1$ and

$$d(x,z) = -\alpha x \quad \text{for } x,z \text{ in } S_1.$$

Then equation (20) may be replaced by its equivalent for two-dimensional flow so that its solution is approximately

$$f_o(x_1,z_1) = -\alpha\sqrt{\frac{1-x_1}{\ldots}}.$$

It may be shown that substitution into equation (23) gives

$$\frac{\partial d_1}{\partial x} = \frac{\alpha}{\pi}\frac{\partial}{\partial x}\int_0^1 \frac{\sqrt{\frac{1-x_1}{x_1}}\,dx_1}{\left(z+\sqrt{z^2+(x-x_1)^2}\right)^2} \tag{24}$$

Equation (22) is replaced by the simpler equation to which it corresponds in lifting line theory. This is

$$\alpha_1 - \beta_1 = \frac{\Gamma_1(z)}{\pi} - \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Gamma'(z_1)}{z_1 - z}\,dz_1 \tag{25}$$

where α_1 and β_1 are the incidence and angle of zero lift respectively corresponding to the displacement $y = d_1(x,z)$ defined by equation (24) and $\Gamma_1(z)$ is the circulation about the lifting line which is supposed to coincide with the Oz axis. It may be shown successively that

$$\alpha_1 - \beta_1 = \frac{2\alpha}{\pi^2}\int_0^1\int_0^1 \frac{\sqrt{\frac{1-x_1}{x_1}}\,(x-t)\sqrt{\frac{x}{1-x}}}{\left[z+\sqrt{z^2+(x-x_1)^2}\right]^2\sqrt{z^2+(x-x_1)^2}}\,dx_1\,dx$$

$$= \frac{4\alpha}{\pi^2}\int_0^1 \frac{\xi^2 F\left(\frac{\pi}{2},\sqrt{1-\xi^2}\right)d\xi}{\left[z+\sqrt{z^2+\xi^2}\right]^2\sqrt{z^2+\xi^2}} \tag{26}$$

where F is the complete elliptic integral of the first kind. In deducing this result the transformation $x - x_1 = \xi$, $x + x_1 = \eta$ is employed.

The expression (26) for $\alpha_1 - \beta_1$ is singular for $z = 0$ and it may be shown that for z small

$$\alpha_1 - \beta_1 = \frac{2\alpha}{\pi^2}\left\{[\log z]^2 + [4 - 3\log 4]\log z\right\} + O(1) \tag{27}$$

Let the solution $\Gamma_1(z)$ of equation (25) with $\alpha_1 - \beta_1$ given by equation (26) be expressed as

$$\Gamma_1(z) = \Gamma_1^{(1)}(z) + \Gamma_1^{(2)}(z) \tag{28}$$

where $\Gamma_1^{(1)}(z)$ is a function that satisfies

$$\frac{\Gamma_1^{(1)}(z)}{\pi} - \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Gamma_1^{(1)\prime}(z_1)}{z_1 - z}\,dz_1 = \begin{cases}\frac{2\alpha}{\pi^2}\left\{[\log z]^2 + [4-3\log 4]\log(z)\right\} + O(1) & \text{for } z \text{ small} \\ O\left(\frac{1}{z^3}\right) & \text{for } z \text{ large}\end{cases} \tag{29}$$

$$= \alpha_1^{(1)} - \beta_1^{(1)} \quad \text{say.}$$

Then $\Gamma_1^{(2)}(z)$ must satisfy

$$\alpha_1 - \beta_1 - \left(\alpha_1^{(1)} - \beta_1^{(1)}\right) = \frac{\Gamma_1^{(2)}(z)}{\pi} - \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Gamma_1^{(2)\prime}(z_1)}{z_1 - z}\,dz_1 \,. \tag{30}$$

The L.H.S. of this equation is not singular for $z = 0$, so that its solution may be obtained by a standard method. $\Gamma_1^{(1)}(z)$ was taken to be

$$\Gamma_1^{(1)}(z) = \begin{cases} a_0 + a_1(z\log z - z) + a_2 z & 0 < z < 1 \\ \frac{a_{-2}}{z^2} + \frac{a_{-3}}{z^3} & z > 1 \end{cases} \tag{31}$$

where the a's were determined by the conditions (29) and the conditions that $\Gamma_1^{(1)}$ and $\Gamma_1^{(1)\prime}$ be continuous at $z = 1$.

Equation (30) was solved by Multhopp's method, see (4), using uniformly spaced pivotal points with spacing 0·2.

Results are given in Figures 2 and 3.

It follows from equation (20) that the total circulation $\Gamma(z)$ is given by

$$\Gamma(z) = \Gamma_0(z) + F^2\Gamma_1(z) + o(F^2)$$

where $\Gamma_0(z) = \pi\alpha$ is the circulation for two-dimensional flow. Thus if $C_L(z)$ denotes the local side force coefficient then

$$C_L(z) = 2\pi\alpha + 2F^2\,\Gamma_1(z) + o(F^2) \tag{32}$$

and the values of $\Gamma_1(z)$ are given in Figure 2.

3.3.2 Case of boat whose draft is small compared to its length.

Consider the case when $h(x) << 1$ and is a monotonic increasing function of x for $0 < x < 1$, and

$$d(x,z) = -\alpha x \qquad x,z \text{ in } S_1 .$$

Then the solution of the slender wing theory approximation to equation (21) is, Robinson and Laurmann (4)

$$f_o(x_1,z_1) = -\frac{\alpha\, h'(x_1)}{\sqrt{1 - \dfrac{z_1^2}{h^2(x_1)}}} \quad . \tag{33}$$

Substitution into equation (23) gives

$$\frac{\partial d_1}{\partial x} = -\frac{\alpha}{\pi} \lim_{y \to 0} \frac{\partial^2}{\partial x \partial y} \int_\infty^z \int_0^1 \int_0^{h(x_1)} \frac{y\, h'(x_1)\; dz_1 dx_1 d\zeta}{[(x-x_1)^2+y^2+(\zeta+z_1)^2]^{3/2} \left[1 - \dfrac{z_1^2}{h^2(x_1)}\right]^{\frac{1}{2}}} \tag{34}$$

The R.H.S. of equation (34) is simplified by introducing a fundamental approximation of lifting line theory. Thus

$$\int_\infty^z \frac{d\zeta}{[(x-x_1)^2+y^2+(\zeta+z_1)^2]^{3/2}} = \frac{1}{(x-x_1)^2+y^2} \left[\frac{\zeta + z_1}{[(x-x_1)^2+y^2+(\zeta+z_1)^2]^{\frac{1}{2}}} \right]_{\zeta=\infty}^{z}$$

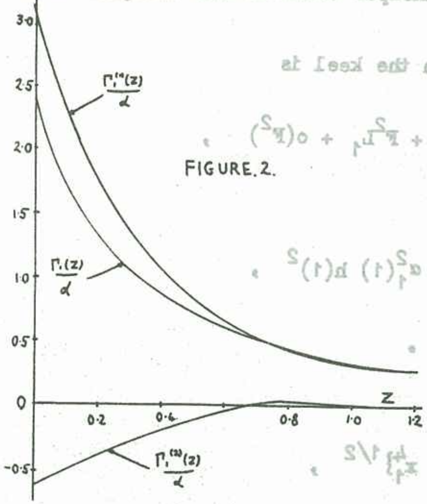
is approximated to by $-\dfrac{1}{(x-x_1)^2+y^2}$.

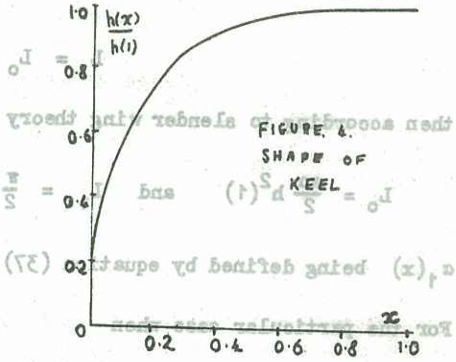
Substituting into equation (34), performing the integration with respect to z_1, and integrating partially with respect to x_1 gives, supposing

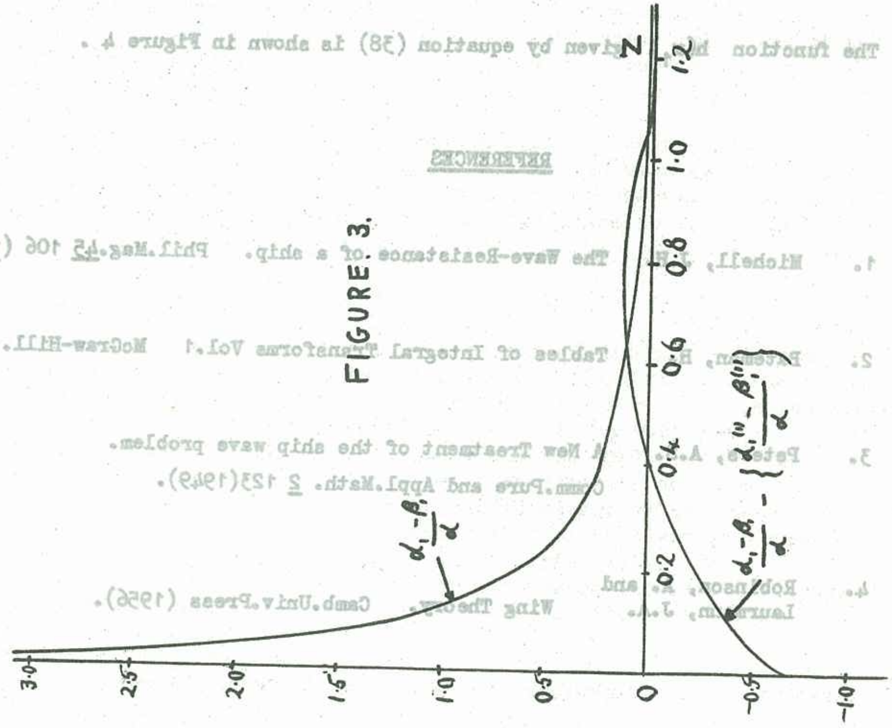
$$h(0) = 0 \quad \text{and} \quad h'(1) = 0 , \tag{35}$$

$$\frac{\partial d_1}{\partial x} = \frac{\alpha}{4} \lim_{y \to 0} \frac{\partial}{\partial x} \int_0^1 \frac{(x_1-x) \dfrac{d^2}{dx_1^2}\{h^2(x_2)\}\, dx_1}{(x_1-x)^2 + y^2}$$

$$= \frac{\alpha}{4} \frac{\partial}{\partial x} P \int_0^1 \frac{\dfrac{d^2}{dx_1^2}\{h^2(x_1)\}\; dx_1}{x_1 - x} , \tag{36}$$







where P denotes that the Cauchy principal value of the integral is to be taken.

If the total side force acting on the keel is

$$L = L_0 + F^2 L_1 + o(F^2) \quad ,$$

then according to slender wing theory

$$L_0 = \frac{\pi\alpha}{2} h^2(1) \quad \text{and} \quad L_1 = \frac{\pi}{2}\alpha_1^2(1)\, h(1)^2 \quad ,$$

$\alpha_1(x)$ being defined by equation (37).

For the particular case when

$$h(x_1) = h(1)\{4x_1 - 6x_1^2 + 4x_1^3 - x_1^4\}^{1/2} \quad , \qquad (38)$$

it is found that

$$L = \frac{\pi}{2}\alpha\, h^2(1)\{1 + 3F^2 h^2(1)\} + o(F^2) \quad . \qquad (39)$$

The function $h(x_1)$ given by equation (38) is shown in Figure 4.

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WIND GENERATION OF WATER WAVES

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ABSTRACT

Water waves are generated by the wind through the direct action of the turbulent pressure field on the water surface, and are amplified by the interaction between the wave motion and the turbulent air flow. The amplification is associated with the presence of a non-zero correlation between the water surface displacement and the air turbulent velocity components. The turbulent velocity field may therefore be decomposed into the mean wind, the mean air wave motion, and the turbulent motion, the mean air wave motion being the part of the field causing the non-zero correlation with the water wave motion. If the water surface has the simple form of a rough nearly-sinusoidal wave train, the mean air wave motion is also sinusoidal of the same wavelength, with a phase and amplitude that vary with height above the surface.

The structure of the air flow is examined using this representation, and an estimate is made of the wave amplification. The estimate is compared with observations of the amplification of a wave train at sea. An experiment is proposed to test this and other wave generation theories.

1. Problem

The air-water interaction can be understood only when the mechanisms are determined relating the turbulent characteristics of the air motion to the form of the water wave motion. Waves are generated by the direct uncoupled action of the turbulent pressure field on the water surface, and are amplified by the coupled interaction between the water wave motion and the turbulent velocity field. The uncoupled mechanism has been investigated by Phillips (1), who found that the turbulent pressure field amplifies certain frequencies preferentially. This mechanism of generation is always present when interaction occurs between wind and water, since atmospheric motions are invariably turbulent.

The important mechanism for water wave amplification by the