where P denotes that the Cauchy principal value of the integral is to be taken.

If the total side force acting on the keel is

$$L = L_0 + F^2 L_1 + o(F^2) \quad ,$$

then according to slender wing theory

$$L_0 = \frac{\pi\alpha}{2} h^2(1) \quad \text{and} \quad L_1 = \frac{\pi}{2} \alpha_1^2(1)\, h(1)^2 \quad ,$$

$\alpha_1(x)$ being defined by equation (37).

For the particular case when

$$h(x_1) = h(1)\{4x_1 - 6x_1^2 + 4x_1^3 - x_1^4\}^{1/2} \quad ,$$

it is found that

$$L = \frac{\pi}{2} \alpha\, h^2(1)\{1 + 3F^2 h^2(1)\} + o(F^2) \quad .$$

The function $h(x_1)$ given by equation (38) is shown in Figure 4.

REFERENCES

1. Michell, J.H. The Wave-Resistance of a ship. Phil.Mag. 45 106

2. Bateman, H. Tables of Integral Transforms Vol.1 McGraw-Hill.

3. Peters, A.S. A New Treatment of the ship wave problem. Comm.Pure and Appl.Math. 2 123(1949).

4. Robinson, A. and Laurmann, J.A. Wing Theory. Camb.Univ.Press (1956).

WIND GENERATION OF WATER WAVES

P.J. Bryant
Mathematics Dept, University of Canterbury, N.Z.

ABSTRACT

Water waves are generated by the wind through the direct action of the turbulent pressure field on the water surface, and are amplified by the interaction between the wave motion and the turbulent air flow. The amplification is associated with the presence of a non-zero correlation between the water surface displacement and the air turbulent velocity components. The turbulent velocity field may therefore be decomposed into the mean wind, the mean air wave motion, and the turbulent motion, the mean air wave motion being the part of the field causing the non-zero correlation with the water wave motion. If the water surface has the simple form of a rough nearly-sinusoidal wave train, the mean air wave motion is also sinusoidal of the same wavelength, with a phase and amplitude that vary with height above the surface.

The structure of the air flow is examined using this representation, and an estimate is made of the wave amplification. The estimate is compared with observations of the amplification of a wave train at sea. An experiment is proposed to test this and other wave generation theories.

1. Problem

The air-water interaction can be understood only when the mechanisms are determined relating the turbulent characteristics of the air motion to the form of the water wave motion. Waves are generated by the direct uncoupled action of the turbulent pressure field on the water surface, and are amplified by the coupled interaction between the water wave motion and the turbulent velocity field. The uncoupled mechanism has been investigated by Phillips (1), who found that the turbulent pressure field amplifies certain frequencies preferentially. This mechanism of generation is always present when interaction occurs between wind and water, since atmospheric motions are invariably turbulent.

The important mechanism for water wave amplification by the wind is the coupled interaction between the water surface displacement field and the air turbulent velocity field. It only acts when water wave motion is already present, when it results in a non-zero correlation being set up between the wave motion and the turbulent air motion. It has been investigated by Miles (2) for laminar air flow, but recent measurements by Snyder and Cox (3) of wave amplification at sea do not agree with Miles's deductions. The coupled interaction is investigated below without making Miles's approximation of laminar air flow, when it is found that the wave amplification predicted does agree with the measurements of Snyder and Cox.

The analysis is simplified by considering a laboratory situation, from which deductions can be made about particular natural situations. The laboratory experiment proposed is based on the wind-water tunnel, a conventional wind tunnel partly filled with water, in which the form of the water surface as well as the character of the turbulent air flow can be regulated upwind of the test section. The situation examined is the amplification along the wind tunnel of an initially sinusoidal water wave train. This choice not only simplifies the analysis, but is probably the best situation for testing experimentally the deductions of the analysis.

2. Interaction

The contribution of the air velocity field to the non-zero correlation with the simplified nearly-sinusoidal displacement field may be identified with a mean motion within the air flow. It is named the mean air wave motion, and has the same wavelength as the water wave train because it is caused by its shape and movement. Its amplitude and phase vary, in general, with height above the mean water surface.

The air velocity field can be expressed as the sum of the mean wind, the mean air wave motion, and the turbulent fluctuations. The mean wind is measured by averaging the velocity field at a point fixed in space. The mean air wave motion is measured either by the air velocity - surface displacement correlation, or by averaging the velocity field in a frame of reference moving with the water wave train. The two measurements of the mean air wave motion give the same result, because the dominant sinusoidal mode in the surface displacement field is time independent in the moving frame of reference.

The turbulent velocity intensities are perturbed periodically by the water wave train, and so also are all other mean velocity products. The intensity perturbations may be measured by the triple correlation between the two appropriate air velocity components and the water surface displacement, or directly by averaging the velocity products in the frame of reference moving with the water wave train. They are negligible in a laminar air flow, and it is their presence in a turbulent air flow that causes the important dynamic differences between wave amplification by laminar and turbulent flows.

The air pressure field can be expressed similarly as the sum of the mean pressure, the mean air wave pressure, and the turbulent pressure fluctuations. The turbulent pressure intensity is also perturbed periodically by the water wave train, as are all mean products of the pressure with itself and with the velocity components.

The axes of reference are chosen with the x-axis in the direction of wave propagation, the z-axis vertically upwards with origin in the mean free surface, and the y-axis completing the orthogonal set. In the proposed experiment, the mean wind and wave propagation directions are the same, and the mean velocity products are independent of the transverse y-coordinate, so this situation is adopted in the analysis. It may be generalised without difficulty to naturally-occurring situations in which the mean wind and wave propagation directions are different. The water surface displacement is taken locally to be a roughened form of the sinusoidal wave train

$$\eta = a \sin k(x - ct) ,$$

where a is the amplitude, k the wave number, and c the phase velocity.

The air velocity, $\underline{u}(x,y,z,t)$, equal to (u,v,w) in the (x,y,z) directions respectively, is divided into the three parts

$$\underline{u} = \underline{U} + \underline{u}_1 + \underline{u}_2.$$

$\underline{U}(x,z)$ is the mean wind $(U,0,W)$, where W is much less than U and is the small mean vertical wind due to the variation in boundary layer thickness. The mean air wave components of velocity, $\underline{u}_1(x,z,t)$, equal to $(u_1,0,w_1)$, are periodic in x and t. They are independent of t if measured in the frame of reference moving with velocity c. The air turbulent velocity fluctuations, $\underline{u}_2(x,y,z,t)$, equal to (u_2,v_2,w_2), have zero mean in both the fixed and moving frames of reference. The air pressure is divided into the three parts

$$p = P + p_1 + p_2,$$

with similar interpretations.

This method of analysis of the air motion is illustrated in figure 1, which shows the distortion by the water wave motion of a free mode in the turbulent air flow. It is to be noted that the free mode is an interpretation of the contribution of the turbulent air flow to the cross-correlation in a given frequency band between points in the flow separated in space and time, and cannot be isolated as a physical entity. It should be drawn with

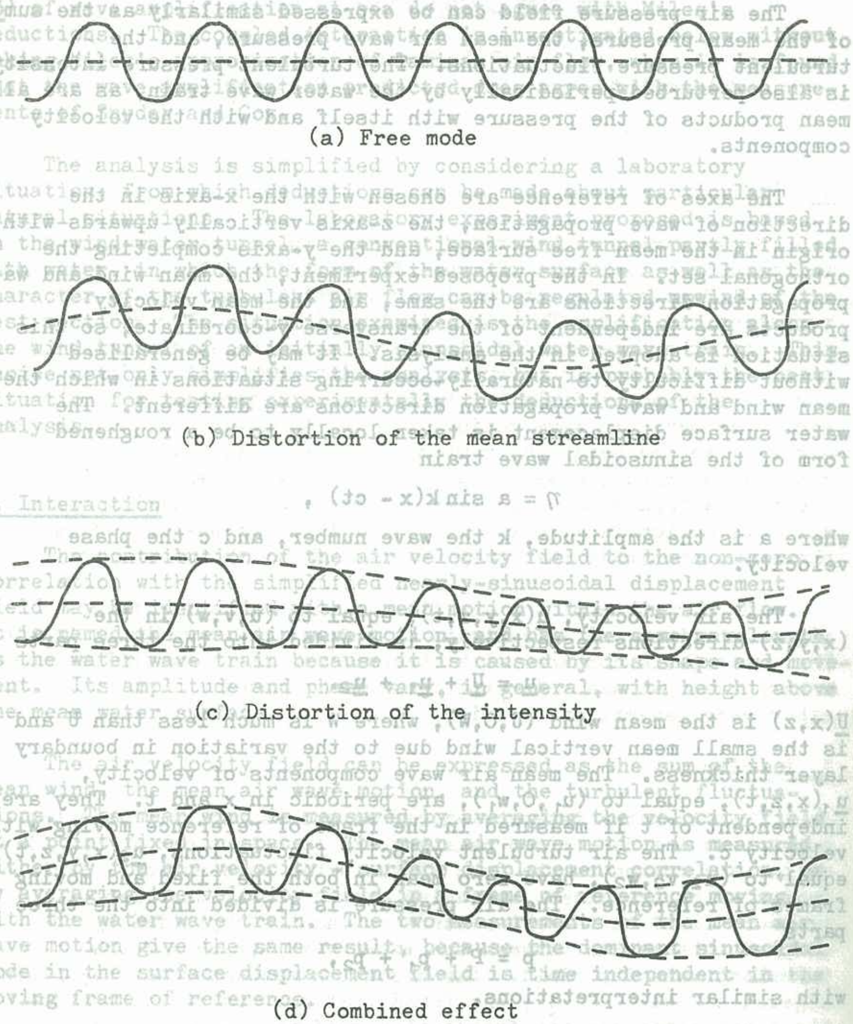

Figure 1. Distortion by the water wave motion of a free mode in the turbulent air flow.

an envelope of similar form to the cross-correlation itself, but for ease of illustration is drawn as a sinusoidal wave train of finite length.

The two dominant types of distortion are shown separately in figures 1(b) and 1(c), and together in figure 1(d). The periodic distortion of the mean streamlines observed in a frame of reference moving with the water wave train is shown in figure 1(b), and is equivalent algebraically to addition of the mean air wave motion to the free mode. The periodic distortion of the intensity of the free mode, again observed in a frame of reference moving with the water wave train, is drawn in figure 1(c). It shows how the intensity perturbation measured in the moving frame varies periodically with the position of measurement. Figure 1(d) is a sketch of the real situation, with both types of distortion occurring together.

3. Equations

The equations of motion for the turbulent air flow can be averaged in both the fixed and moving frames of reference. The fixed frame means are the usual equations for the mean motion in a turbulent shear flow, discussed for example by Townsend (4).

The equations for the mean air wave motion can be obtained by subtracting the mean of the equations of the air motion in a fixed frame of reference from the mean of the equations of the air motion in a frame of reference moving with the water wave train. The notation $\overline{f}$ denotes the mean of f with respect to time in the fixed frame, while $\overline{f}'$ denotes the mean of f with respect to time in the moving frame. The difference $\overline{f}' - \overline{f}$ is periodic in x with zero mean. The equations for the mean air wave motion are found to be

$$(U-c)\frac{\partial u_1}{\partial x} + w_1\frac{\partial U}{\partial z} + \frac{\partial}{\partial x}(\overline{u_2^2}' - \overline{u_2^2}) + \frac{\partial}{\partial z}(\overline{u_2 w_2}' - \overline{u_2 w_2}) = -\frac{1}{\rho}\frac{\partial p_1}{\partial x},$$

and

$$(U-c)\frac{\partial w_1}{\partial x} + \frac{\partial}{\partial x}(\overline{u_2 w_2}' - \overline{u_2 w_2}) + \frac{\partial}{\partial z}(\overline{w_2^2}' - \overline{w_2^2}) = -\frac{1}{\rho}\frac{\partial p_1}{\partial z}.$$

If the perturbations in the turbulent velocity products could be expressed in terms of the mean air wave motion, these equations could be solved exactly for the flow. This is not possible since the required relationships must be determined experimentally, so all deductions made from these equations are qualitative only.

Similar manipulation of the continuity equation gives

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0.$$

4. Solution

The streamlines of the perturbed mean air motion are sketched

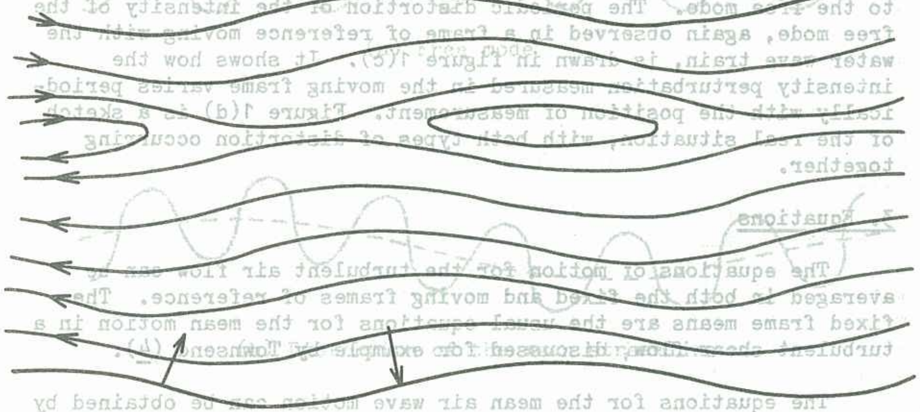

Figure 2. The streamlines of the perturbed mean air motion as seen by an observer moving with the water wave train. The arrows on the water surface indicate the direction of the perturbed mean air pressure on the water surface. The region with the loop structure is the critical layer, where the mean wind velocity is equal to the phase velocity of the water wave train.

in figure 2. The critical layer, the region where $U(z) - c$ is zero, is shown with a loop structure, a property that can be demonstrated by a kinematic argument. The phase of the mean air wave motion increases with increasing height. This causes the streamlines to be closer together ahead of the crests and further apart behind the crests than if the phase were constant, and is associated with a pressure deficit ahead of the crests and a pressure excess behind the crests as is indicated by the two arrows in the figure. This component of pressure causes wave amplification.

It has been proposed by Hasselmann (5) that the spectral energy density $F(\underline{k};\underline{r},t)$ of a general surface displacement field satisfies the equation

$$\partial F/\partial t + \underline{c_g}(\underline{k}).\,\mathrm{grad}\,F = \alpha(\underline{k};\underline{r},t) + \beta(\underline{k};\underline{r},t)F - 4\nu_T k^2 F + N,$$

where $\underline{c_g}$ is the group velocity corresponding to the wave number $\underline{k}$. The first two terms on the right hand side are a linear expansion of the rate of working by the turbulent air flow, the first term arising through uncoupled mechanisms and the second through linear coupled mechanisms of energy transfer. The third term describes the rate of loss of energy by dissipation under turbulent action, and the fourth covers all energy loss or transfer by non-linear processes.

The mean air wave pressure, p_1, is the perturbation to the mean pressure observed in the frame of reference moving with the water wave train. It causes a coupled energy transfer to the water surface and so contributes to the β-term in Hasselmann's equation. Solution for p_1 from the equations of motion above, and substitution in Hasselmann's equation, shows that

$$\beta \doteqdot 2\rho_a \overline{u_2^2} k^2 c/\rho_w g,$$

where ρ_a, ρ_w are the densities of air and water, $\overline{u_2^2}$ is measured near the water surface, k is the wave number and c the phase velocity of the water wave train.

Snyder and Cox (3) measured the amplification of an ocean wave train of wavelength 17m propagating in deep water. Their data show that β is given by

$$\beta \doteqdot 2 \times 10^{-5} U^2,$$

where U, the wind speed at a height of 10m, is measured in m/sec. The estimate for β from above is

$$\beta \doteqdot 1.5 \times 10^{-4} \overline{u_2^2}.$$

Theory and observation are in order of magnitude agreement if the missing measurement is

$$\overline{u_2^2}/U^2 \doteqdot 0.1\,.$$

This figure lies within the range of measurements of this ratio in natural conditions, (for example, by Cramer (6)).

5. Discussion

The agreement between theory and experiment is encouraging, but a proper test must await a controlled laboratory experiment in which the wind structure and the wave amplification are measured together.

The mechanism of amplification on a general surface displacement field is the same in essence as that described above for the simplified laboratory situation. The mean velocity and the turbulent intensities are perturbed by the water motion, each perturbation being apparent as a non-zero correlation between the water motion and the appropriate parameter of the air motion. These perturbations lead to a pressure component on the water surface in phase with the wave slope, which in turn amplifies the water wave motion.

The uncoupled action of the turbulent pressure field on the water surface is to generate waves preferentially in a single frequency band at each forward angle to the wind, (Phillips(1)). The combined effect of the two mechanisms, uncoupled and coupled, is therefore to amplify preferentially the frequency band propagating in the direction of the mean wind. This is the probable explanation for the rapid growth of a nearly-sinusoidal wave motion in the direction of the mean wind, when gusts of wind last for more than a few moments on a water surface initially at rest. The coupling with the turbulent velocity field amplifies exponentially the frequency band chosen by the turbulent pressure field.

References

(1) Phillips, O.M. On the generation of waves by turbulent wind. J. Fluid Mech., 2, 417, (1957).

(2) Miles, J.W. On the generation of surface waves by shear flows. J. Fluid Mech., 3, 185, (1957).

(3) Snyder, R., and Cox, C.S. A field study of the wind generation of ocean waves. (To be published).

(4) Townsend, A.A. The structure of turbulent shear flow. C.U.P. (1956).

(5) Hasselmann, K. Grundgleichungen der Seegangsvoraussage. Schiffstechnik Bd. 7, 191, (1960).

(6) Cramer, H.E. Measurements of turbulence structure near the ground. Advances in Geophysics, 6, 75, (1959).

ON CONVECTION IN A POROUS MEDIUM

By

A. McNabb

Applied Mathematics Division, D.S.I.R., Wellington, N.Z.

Abstract.

Boundary layer equations are derived for convective flow of fluid over a hot plate on the bed of a semi infinite porous medium. An estimate for the heat convected away as a function of the temperature of the hot plate is used to evaluate the rate of cooling of a lava lake under a water saturated porous medium.

1. Introduction.

Geothermal activity at Wairakei originates in a bed of porous water saturated volcanic debris about two and a half kilometers deep. This activity appears to be fed by a jet of hot water with a cross sectional area of two to three square kilometers and central temperatures up to two hundred and fifty degree centigrade, convected from the basement of the system.

In this paper we examine the possibility that this jet is generated on the surface of an old magma lake cooling by conduction of heat through a solid crust. We imagine water flowing in a boundary layer along the bottom and being heated as it passes over the hot crust. The bulk of the fluid above, is at a constant temperature and motionless and hence sustaining a hydrostatic pressure distribution. Since the heated fluid in the boundary layer is diminished in density the pressure on the bottom is lower than hydrostatic by a margin that increases with the thickness of this hot layer. This produces a pressure gradient along the bottom driving fluid in the direction of thickening of this layer towards a stagnation point where it leaves the lake and rises toward the surface.

We consider first a simplified problem involving steady convection in an infinite homogeneous porous medium in the half plane $z > 0$, (z measures height above the plane horizontal impermeable floor of the region). The convection current is generated by a hot plate of radius a in the plane of the floor at a constant temperature T_1 which is greater than the temperature T_0 of the medium sufficiently far from the plate. This problem avoids any interaction between the fluid flow over the hot plate and that induced by the jet impinging on an upper boundary or free surface.