5. The method proposed needs to be checked against experimental data for various cases to find out its efficiency.

The author is investigating this method further and will be reporting the results of his work in a future paper.

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# THE USE OF THREE FIXED WAVE PROBES TO DETERMINE THE REFLECTED AND INCIDENT COMPONENTS OF A WAVE MOTION

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## SYNOPSIS

In a wave flume a wave motion generally has incident and reflected components.

An expression for obtaining the incident and reflected components of the partial clapotis by measuring the wave height at a point chosen at random, and at two other points each one-eighth of a wave length from the first point, is developed.

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In two dimensional work in a wave flume the propagated wave from the generating paddle may be affected by reflection from a beach so that measurement of the incident wave amplitude a is fairly difficult without recourse to a slowly moving probe traversing the length of the flume.

In what follows it is shown that by using the envelope value of wave-heights at three fixed stations, one-eighth of a wave-length apart, the incident wave height, the reflected wave height and the reflection coefficient may be readily determined.

Consider a wave motion propagated in the positive $x$ direction in constant depth with period $T$. Using the first order small amplitude wave theory, the wave-length $L$, is given by $L = \frac{gT^2}{2\pi} \tanh mh$ and the instantaneous surface elevation $y$ at any position $x$ at any time $t$ by $y = a \sin(mx - kt + \phi)$ where $k = \frac{2\pi}{T}$, $m = \frac{2\pi}{L}$, $a$ is the maximum amplitude from either crest or trough to still-water level, and $\phi$ is an arbitrary phase relationship measured from the origin of $(mx - kt)$.

Consider a negative wave, of the same period $T$ and hence having the same wave number, $m$, and the same wave angular frequency $k$ as the incident wave. Its surface elevation $y'$ is given by $y' = a' \sin(mx + kt + \phi')$.

Superposition of these two linear systems gives

$$Y = y + y' = a \sin(mx - kt + \phi) + a' \sin(mx + kt + \phi')$$

If the origin of $x$ is so chosen that $\phi = 0$

$$Y = a \sin(mx - kt) + a' \sin(mx + kt + \phi'), \text{ i.e.,}$$

$$Y = a \sin(mx - kt) + a' \cos\phi' \sin(mx + kt) + a' \sin\phi \cos(mx + kt)$$

If the negative wave is generated by a partial or total reflection of the positive wave at the same barrier, then $\frac{a'}{a} = R$ the reflection coefficient, $(R \leq 1)$.

If $R$ is unity, that is perfect reflection, $a = a'$. At an impermeable boundary there is an antinode $\phi' = \pi, 3\pi, 5\pi$, etc.

$$Y = -2a \sin kt \cos mx \qquad \ldots (1)$$

and a standing wave system is produced.

If instead of a total reflection only partial reflection takes place, $a > a'$. If in addition the phase angle $\phi'$ is assumed to be $\pi$ as before, then

$$Y = a \sin(mx - kt) + a' \sin(mx + kt + \pi),$$

i.e., $Y = a \sin(mx - kt) - a' \sin(mx + kt)$,

i.e., $Y = (a + a') \sin(mx - kt) - a'(\sin(mx - kt) + \sin(mx + kt))$



where the first term is progressive and the second term standing; or $Y = (a - a') \sin mx \cos kt - (a + a') \cos mx \sin kt \qquad \ldots (2)$

Equations (1) and (2) then correspond to equation (2) in the paper by C. Carry (1) and describe curves such as those in La Houille Blanche (2).

The two extreme amplitudes of the envelope of amplitudes are $(a + a')$ and $(a - a')$ at distances $\frac{L}{4}$ apart.

Substituting $Ra$ for $a'$

$$Y = a(1 - R) \sin mx \cos kt - a(1 + R) \cos mx \sin kt.$$

At any particular value of $x$ let

$\cos mx = c$, $\sin mx = \sqrt{1 - c^2}$.

Then $Y$ at a particular $x$ at any time $t$ is given by

$$Y = a(1 - R)\sqrt{1 - c^2} \cos kt - a(1 + R)\, c \sin kt.$$

The amplitude of the motion at this particular $x$ varies over a period and its maximum or minimum value is given when $\frac{\partial Y}{\partial t} = 0$. That is, when

$$0 = ak\left((R - 1)\sqrt{1 - c^2} \sin kt - c(R + 1) \cos kt\right)$$

that is, when

$$\sin kt = \frac{c(R + 1)}{\sqrt{(1 - c^2)(R - 1)^2 + c^2(R + 1)^2}}$$

$$\cos kt = \frac{\sqrt{1 - c^2}\,(R - 1)}{\sqrt{(1 - c^2)(R - 1)^2 + c^2(R + 1)^2}}$$

Thus the maximum or minimum amplitude of $Y$ at any $x$ is given by

$$Y\,(\text{max. or min.}) = -a\sqrt{(R - 1)^2 + 4c^2R}$$

If the envelope value of $Y$ at any initial position $x = A$ is denoted by $a_A$, then

$$\cos \frac{2\pi A}{L} = c.$$

$$a_A = -a\sqrt{(R - 1)^2 + 4c^2R} \qquad \ldots (3)$$

At a position $x = B$, distance $L/8$ from $A$ in the $x$ direction

$$\cos \frac{2\pi}{L}(A + L/8) = \frac{1}{\sqrt{2}}\left(c - \sqrt{1 - c^2}\right)$$

$$a_B = -a\sqrt{(R^2 + 1) - 4Rc\sqrt{1 - c^2}} \qquad \ldots (4)$$

At a further point $x = C$, distance L/4 from A

$$\cos \frac{2\pi}{L}\left(A + \frac{L}{4}\right) = -\sqrt{1 - c^2}$$

$$a_C = -a\sqrt{(R + 1)^2 - 4Rc^2} \quad \ldots (5)$$

Thus we have three equations and three unknowns, a, R and c. These equations can be re-written so that from any three known measurements, $a_A$, $a_B$ and $a_C$ taken L/8 apart, a and R can be determined. (c is of no particular value since it only determines the origin of x).

Combining equations ( 3 ) and ( 5 ),

$$\frac{a_A^2 + a_C^2}{a^2} = 2(R^2 + 1)$$

$$\therefore 1 + R^2 = \frac{a_A^2 + a_C^2}{2a^2} \quad \ldots (6)$$

and

$$\frac{a_A^2 - a_C^2}{a^2} = 8c^2R - 4R$$

$$\therefore 8Rc^2 = 4R + \frac{a_A^2 - a_C^2}{a^2}$$

$$c^2 = \frac{1}{2} + \frac{a_A^2 - a_C^2}{8Ra^2}$$

$$\text{and } (4Rc)^2 = 2R(8Rc^2) = 8R^2 + 2R\,\frac{a_A^2 - a_C^2}{a^2}$$

$$\therefore (4Rc)^2(1 - c^2) = \left(8R^2 + 2R\,\frac{a_A^2 - a_C^2}{a^2}\right)\left(\frac{1}{2} - \frac{a_A^2 - a_C^2}{8Ra^2}\right)$$

$$= 4R^2 - \frac{1}{4}\left(\frac{a_A^2 - a_C^2}{a^2}\right)^2 \quad \ldots (7)$$

Substituting equations (6) and (7) in (4)

$$\frac{a_B^2}{a^2} = \frac{a_A^2 + a_C^2}{2a^2} - \sqrt{4\left(\frac{a_A^2 + a_C^2}{2a^2} - 1\right) - \frac{1}{4}\left(\frac{a_A^2 - a_C^2}{a^2}\right)^2}$$



$$\therefore 4\left(\frac{a_A^2 + a_C^2}{2a^2} - 1\right) - \frac{1}{4}\left(\frac{a_A^2 - a_C^2}{a^2}\right)^2 = \left(\frac{a_A^2 + a_C^2 - 2a_B^2}{2a^2}\right)^2$$

$$\therefore 16a^4 - 8a^2(a_A^2 + a_C^2) + (a_A^2 - a_C^2)^2 + (a_A^2 + a_C^2 - 2a_B^2)^2 = 0$$

$$\therefore a^2 = \frac{1}{4}\left(a_A^2 + a_C^2 \pm \sqrt{(a_A^2 + a_C^2)^2 - (a_A^2 - a_C^2)^2 - (a_A^2 + a_C^2 - 2a_B^2)^2}\right)$$

For the negative square root to be applicable it must be possible that

$4a^2 < a_A^2 + a_C^2$.

But from equation ( 6 )

$a_A^2 + a_C^2 = 2(R^2 + 1)a^2$, that is,

$4a^2 < 2(R^2 + 1)a^2$,

i.e., $R > 1$, which is impossible.

Thus,

$$a = \frac{1}{2}\left(a_A^2 + a_C^2 + \sqrt{4a_B^2(a_A^2 + a_C^2 - a_B^2) - (a_A^2 - a_C^2)^2}\right)^{\frac{1}{2}}$$

$$\text{and } R = \sqrt{\frac{a_A^2 + a_C^2}{2a^2} - 1}$$

Using these expressions the incident wave amplitude a, and the reflection coefficient R may be determined from three measurements of the amplitude of the wave height envelope at points L/8 apart.

Experimental results obtained to date support this theoretical development but are too few to be statistically significant. However, the standard deviation of wave height measurements about a mean has been found to be reduced consistently using the measuring technique and expressions developed above.

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