

On Modelling Velocity/Pressure-Gradient Correlations in Higher-Order RANS Statistical Closures

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Abstract

When the Gram-Charlier series expansions for the velocity correlations are used to close the set of Reynolds-Averaged Navier-Stokes (RANS) equations, no assumption on Gaussian turbulence is invoked and no unknown model coefficients are introduced into the modelled equations. In such a way, this closure procedure reduces the modelling uncertainty of fourth-order RANS closures (FORANS). Models are still required for the interaction of turbulent velocity and pressure fluctuation fields and dissipative processes. The current paper describes new linear models for the second-, third-, and fourth-order velocity/pressure-gradient correlations applicable to two-dimensional incompressible turbulent wall-bounded flows. DNS data for high-order statistics in two-dimensional (2D) fully-developed channel flow and zero-pressure gradient boundary layer over a flat plate are used to verify the model expressions.

Introduction

To improve the performance of RANS turbulence models, one has to improve the accuracy of models for three physical processes: turbulent diffusion, interaction of turbulent pressure and velocity fluctuation fields, and dissipative processes. In one- and two-equation RANS models, all turbulence effects are modelled based on empirical/intuitive considerations rather than on physics-based assumptions about the flow dynamics. This class of models is the least accurate and is not considered in our current study.

The accuracy of modelling the turbulent diffusion depends on the order of a statistical closure chosen as a basis for a RANS model. In second-order closures (or Reynolds-stress transport models), gradients of third-order velocity moments that describe the turbulent diffusion in the Reynolds-stress transport (RST) equations are usually modelled using the semi-empirical generalized gradient-diffusion hypothesis [2,10]. This hypothesis is not derived from the analysis of general physical properties of a turbulent flow field. Hence, models based on this hypothesis or of similar kind (see, for example, [4]) are not physics-based.

In third-order closures, a model for turbulent diffusion can be derived by assuming the quasi-Gaussian turbulence. That is, all velocity moments of the fourth- and higher-orders can be determined from the Gaussian distribution of the probability density function (PDF) of the turbulent velocity field (Millionshtchikov's hypothesis of quasnormality [9]). However, a turbulent velocity field is generally non-Gaussian [14].

A rigorous procedure for closing the set of RANS equations without assuming Gaussian correlations was suggested in [6] for FORANS and higher-order closures. In such expansions, a non-

Gaussian PDF is given in the form of a series in Hermite polynomials with respect to the Gaussian distribution. The Gram-Charlier series expansions are used to represent fifth- and higher-order velocity moments in terms of lower-order velocity moments without unknown model coefficients [3]. The applicability of Gram-Charlier series expansions was successfully tested experimentally in several flows (see reviews in [3,11]). DNS data for high-order statistics were successfully used in [5,11] to validate the closing procedure in a 2D zero-pressure gradient boundary layer over a flat plate and in a 2D fully-developed channel flow. Overall, available evidence provides a sufficient basis for choosing Gram-Charlier series expansions to accurately model the turbulent diffusion in transport equations for fourth-order velocity moments. No modelling is required for turbulent diffusion in transport equations for second- and third-order velocity moments in FORANS closures.

The current paper addresses modelling the interaction of turbulent velocity and pressure fluctuation fields, one of the two remaining physical processes in FORANS equations to model. In transport equations for velocity moments, this process is represented by the velocity/pressure-gradient correlations of different orders.

Models for Velocity/Pressure-Gradient Correlations

The correlations of interest in FORANS equations are $\langle u_i p_{,j} / \rho \rangle$, $\langle u_i u_k p_{,j} / \rho \rangle$, and $\langle u_i u_k u_l p_{,j} / \rho \rangle$. Hereafter, Cartesian notations are used, u and p are velocity and pressure fluctuations, ρ is the density, $f_{i,j} = \partial f_i / \partial x_j$. Initial ideas for modelling such correlations were developed in [1], but no model suitable for practical applications was proposed at that time. Starting from [13], the modelling effort was shifted towards relevant, but different pressure-containing correlations: pressure/velocity-gradient (pressure-strain) correlations $\langle p u_{i,j} / \rho \rangle$.

Although it simplifies a mathematical formulation of the problem, this approach limits the ability of RANS models to adequately describe turbulent flows. The substitution of velocity/pressure-gradient correlations with pressure-strain correlations is only valid in homogeneous turbulence. Moreover, this substitution is of no use for modelling higher-order correlations.

Models for $\langle u_i p_{,j} / \rho \rangle$ and for the excess factor were proposed in [7,12]. A review of models for $\langle u_i u_k p_{,j} / \rho \rangle$ can be found in [8]. No integrated approach was previously proposed for modelling second, third-, and fourth-order correlations. Further, the number of model coefficients was an issue for some of the earlier modelling attempts. In the current paper, a new integrated approach is proposed for modelling velocity/pressure-gradient

correlations of different orders, based on separating the rapid and slowly decaying correlations.

The exact integral-differential expressions (without the surface integrals) for velocity/pressure-gradient correlations are:

$$\begin{aligned}
-\frac{1}{\rho} \langle p_{,j} u_i \rangle &= -\frac{1}{2\pi} \iiint [U'_{m,n} \langle u'_n u_i \rangle'_{,j}]'_{,j} \frac{1}{r} dV' \\
&\quad - \frac{1}{4\pi} \iiint \langle u'_m u'_n u_i \rangle'_{,mnj} \frac{1}{r} dV', \\
-\frac{1}{\rho} \langle p_{,j} u_i u_k \rangle &= -\frac{1}{2\pi} \iiint [U'_{m,n} \langle u'_n u_i u_k \rangle'_{,m}]'_{,j} \frac{1}{r} dV' \\
&\quad - \frac{1}{4\pi} \iiint [\langle u'_m u'_n u_i u_k \rangle - \langle u'_m u'_n \rangle \langle u_i u_k \rangle]'_{,mnj} \frac{1}{r} dV', \\
-\frac{1}{\rho} \langle p_{,j} u_i u_k u_l \rangle &= -\frac{1}{2\pi} \iiint [U'_{m,n} \langle u'_n u_i u_k u_l \rangle'_{,m}]'_{,j} \frac{1}{r} dV' \\
&\quad - \frac{1}{4\pi} \iiint [\langle u'_m u'_n u_i u_k u_l \rangle - \langle u'_m u'_n \rangle \langle u_i u_k u_l \rangle]'_{,mnj} \frac{1}{r} dV',
\end{aligned} \tag{2}$$

[1]. Here, “ ’ ” above a flow variable indicates that it should be evaluated at a point Y' with coordinates x'_i , which ranges over the region of the flow; r is the distance from Y' to the point Y with coordinates x_i ; dV' and dS' are the volume and surface elements, respectively; and $\partial/\partial n'$ denotes the normal derivative. The velocity/pressure-gradient correlations on the left side of (2) are evaluated at point Y , whereas all derivatives on the right side are taken at Y' . Terms with the mean velocity gradients are usually called “rapid”. The second integrals on the right side of (2) are called “slow”.

An integrated approach to modelling “slow” terms in (2) was proposed by the authors in [11]. Models were derived by analysing tensor properties of the integrals. However, due to the scope of the paper and the length of derivations, all derivations are omitted here. Only models in their current form are reported:

$$\begin{aligned}
\Pi_{ij}^{(s)} &= -\frac{1}{\rho} (\langle p_{,j} u_i \rangle^{(s)} + \langle p_{,i} u_j \rangle^{(s)}) = B_1 \langle u_m u_i u_j \rangle_{,m}, \\
\Pi_{ijk}^{(s)} &= -\frac{1}{\rho} (\langle p_{,j} u_i u_k \rangle^{(s)} + \langle p_{,i} u_j u_k \rangle^{(s)} + \langle p_{,k} u_i u_j \rangle^{(s)}) \\
&= B_1 \langle u_m u_i u_j u_k \rangle_{,m} - B_2 \begin{pmatrix} \langle u_i u_k \rangle \langle u_m u_j \rangle_{,m} \\ + \langle u_j u_k \rangle \langle u_m u_i \rangle_{,m} \\ + \langle u_i u_j \rangle \langle u_m u_k \rangle_{,m} \end{pmatrix}, \\
\Pi_{ijkl}^{(s)} &= B_1 \langle u_m u_i u_j u_k u_l \rangle_{,m} \\
&\quad - B_2 \begin{pmatrix} \langle u_i u_k u_l \rangle \langle u_m u_j \rangle_{,m} + \langle u_j u_k u_l \rangle \langle u_m u_i \rangle_{,m} \\ + \langle u_i u_j u_l \rangle \langle u_m u_k \rangle_{,m} + \langle u_i u_j u_k \rangle \langle u_m u_l \rangle_{,m} \end{pmatrix}.
\end{aligned} \tag{3}$$

In (3), tensors Π represent terms describing the interaction of turbulent pressure and velocity fluctuation fields in FORANS equations. Expressions (3) differ slightly from those in [11] where results were obtained at $B_2 = 0$. In [11], good agreement between the model and DNS profiles was observed in wall-bounded flows at $y_+ > 10$ with $B_2 = 0$, but as discussed in the *Results* section, non-zero B_2 is essential for accurate modelling of Π to the wall. The parameter y_+ is defined as $u_\tau y / \nu$, where u_τ is the friction velocity, ν is the kinematic viscosity, and y is the distance from the wall.

The first terms on the right side of (3) can be linked to terms describing turbulent diffusion, $\mathbf{D}^{(T)}$, in RANS equations. Similarly, the second terms can be related to the production terms by turbulence, $\mathbf{P}^{(T)}$. Then, (3) can be re-written as

$$\begin{aligned}
\Pi_{ij}^{(s)} &= -B_1 D_{ij}^{(T)}, \quad \Pi_{ijk}^{(s)} = -B_1 D_{ijk}^{(T)} - B_2 P_{ijk}^{(T)}, \\
\Pi_{ijkl}^{(s)} &= -B_1 D_{ijkl}^{(T)} - B_2 P_{ijkl}^{(T)}.
\end{aligned} \tag{4}$$

Tensor-invariant models proposed in [11] for the “rapid” terms in (2) were developed under the assumption of the pressure fluctuation dependence on the local mean velocity gradients [1]. This assumption holds true when the mean velocity gradients vary more slowly than the two-point velocity correlations within the volume determined by the two-point velocity correlation length scale. Then, the first integrals on the right side of (2) can be simplified. For $\langle u_i p_{,j} / \rho \rangle$, for example, it yields

$$\begin{aligned}
-\frac{1}{\rho} \langle p_{,j} u_i \rangle^{(r)} &= -\frac{1}{2\pi} \iiint [U'_{m,n} \langle u'_n u_i \rangle'_{,j}]'_{,j} \frac{1}{r} dV' \\
&\approx -\frac{1}{2\pi} U_{m,n} \iiint \langle u'_n u_i \rangle'_{,mj} \frac{1}{r} dV'.
\end{aligned} \tag{5}$$

This assumption is violated in the wall proximity and is a cause of the discrepancy of the model Π -profiles in that region in [11]. Therefore, a new integrated approach to modelling “rapid” terms was suggested in [11] based on the analysis of the DNS budgets of transport equations and a relevance of different budget terms to modelling velocity/pressure-gradient correlations. Here, updated model expressions for the Π -components in 2D wall-bounded flows are given that further clarify a mechanism of the energy transfer between velocity moments that involve turbulence production by the mean flow field and the interaction of turbulent velocity and pressure fluctuation field. The “rapid” parts are linear functions of the production terms, but the mechanism is more complex even for Reynolds stresses than thought earlier. Below, the model expressions for “rapid” and “slow” terms are combined together for each Π -component as the coefficient values are interdependent for some correlations:

$$\begin{aligned}
\Pi_{xx} &= -0.1D_{xx}^{(T)} + 0.02P_{xx} + P_{xy}, \quad \Pi_{xy} = -0.5D_{xy}^{(T)} - 0.02P_{xx} - P_{xy}, \\
\Pi_{yy} &= -0.5D_{yy}^{(T)} - 0.025P_{xx} - 0.45P_{xy}, \\
\Pi_{zz} &= -0.5D_{zz}^{(T)} + 0.025P_{xx} - 0.55P_{xy}, \\
\Pi_{xxx} &= -0.1D_{xxx}^{(T)} - 0.3P_{xxx}^{(T)} - 0.05P_{xxx} + 2.5P_{xxy}, \\
\Pi_{xxy} &= -0.1D_{xxy}^{(T)} + 0.05P_{xxy}^{(T)} - 1.1P_{xxy} + 0.8P_{xyy}, \\
\Pi_{xyy} &= -0.5D_{xyy}^{(T)} + 0.2P_{xyy}^{(T)} + 0.1P_{xyy}, \\
\Pi_{yyy} &= -0.5D_{yyy}^{(T)} - 0.4P_{yyy}^{(T)} - 0.9P_{yyy}^{(T)} - 0.25P_{xxy} - 2P_{xyy}, \\
\Pi_{xxxx} &= -0.1D_{xxxx}^{(T)} + 0.08P_{xxxx} + 1.6P_{xxyy}, \\
\Pi_{xxyy} &= -0.5D_{xxyy}^{(T)} + 0.4P_{xxyy}^{(T)} - 0.65P_{xxyy} + 0.5P_{xyyy}, \\
\Pi_{xyyy} &= -0.5D_{xyyy}^{(T)} + 0.8P_{xyyy}^{(T)} - 0.35P_{xyyy} + 0.15P_{xyyy}, \\
\Pi_{xyyy} &= -0.5D_{xyyy}^{(T)} + 1.6P_{xyyy}^{(T)} + 0.15P_{xyyy} - 0.28P_{xyyy}, \\
\Pi_{yyyy} &= -0.5D_{yyyy}^{(T)} + 3.2P_{yyyy}^{(T)} - 0.22P_{xyyy} - 1.5P_{xyyy}.
\end{aligned} \tag{6}$$

In (6), x , y , and z are streamwise, normal-to-wall, and spanwise directions respectively; \mathbf{P} is the tensor of production by the mean velocity field. The components of tensors are not shown here due to the scope of the papers, but they are straightforward to derive in a standard way for RANS equations.

The value of B_1 is 0.1 for the correlations in the streamwise direction and for Π_{xxy} . The behaviour of Π_{yy} and Π_{zz} is less sensitive to this parameter: it does not change dramatically for the values between 0 and 0.5 for both correlations. One of the conclusions to be made from (6) is that correlations between velocity and pressure-gradient fluctuations in the streamwise direction have a weaker link to the turbulent diffusion processes than correlations of the fluctuations in other directions. Notice also that velocity/pressure-gradient correlations participate in the energy exchange between velocity moments through the production due to mean shear in the entire plane, not simply along a single direction. Expressions (6) should be considered as a proof of concept rather than the final model formulation.

Results

Expressions (6) were validated against DNS data [5] in a 2D fully-developed channel. In a zero-pressure gradient boundary layer over a flat plate, the budget terms are currently available only for the second-order correlations. The results for this flow will be reported at the conference. Based on our previous results [11], they are expected to be as good as in a channel flow.

In figure 1, model profiles obtained from (6) using the DNS data for the budget terms in the transport equations for corresponding velocity moments are shown by solid lines. DNS data for the velocity/pressure-gradient correlations are shown by black circles. The agreement between the models and the DNS data is good everywhere in the flow including the near-wall area.

Although some improvement is possible and more study is required to finalize the model expressions, the results obtained demonstrate that velocity/pressure-gradient correlations can be modelled up to the wall in terms of turbulent diffusion and production terms without introducing any empirical/damping functions.

Conclusions

The paper presents new linear models for second-, third-, and fourth-order velocity/pressure-gradient correlations in transport equations of FORANS closures applicable to 2D wall-bounded flows. Models for the “slow” parts of exact integral-differential expressions for the correlations are tensor-invariant and applicable to an incompressible turbulent flow in any geometry. Tensor-invariant models for the “rapid” terms were developed as well and presented in [11], but their validation in wall-bounded flows revealed their deficiency at $y_+ < 50$. Thus, to improve the description in the near-wall area, the analysis of the DNS budgets of transport equations for velocity moments and a relevance of different budget terms to modelling velocity/pressure-gradient correlations was conducted using DNS data [5]. In the current paper, models for the “rapid” terms based on that analysis are given. The models reveal new physical mechanisms of the energy exchange between different velocity moments. That is, velocity/pressure-gradient correlations participate in the energy exchange between velocity moments through the mechanism of production due to mean shear in the entire plane, not simply along a single direction.

Comparison of the model profiles for velocity moments with DNS data in a fully-developed channel [9] shows good agreement for all correlations everywhere in a flow including the near-wall area. Similar results are expected in a zero-pressure gradient boundary layer over a flat plate at different Reynolds numbers as preliminary study [8] demonstrated. They will be presented at the conference.

The results presented in the paper demonstrate that velocity/pressure-gradient correlations can be modelled up to the

wall in terms of turbulent diffusion and production terms without introducing any empirical/damping functions.

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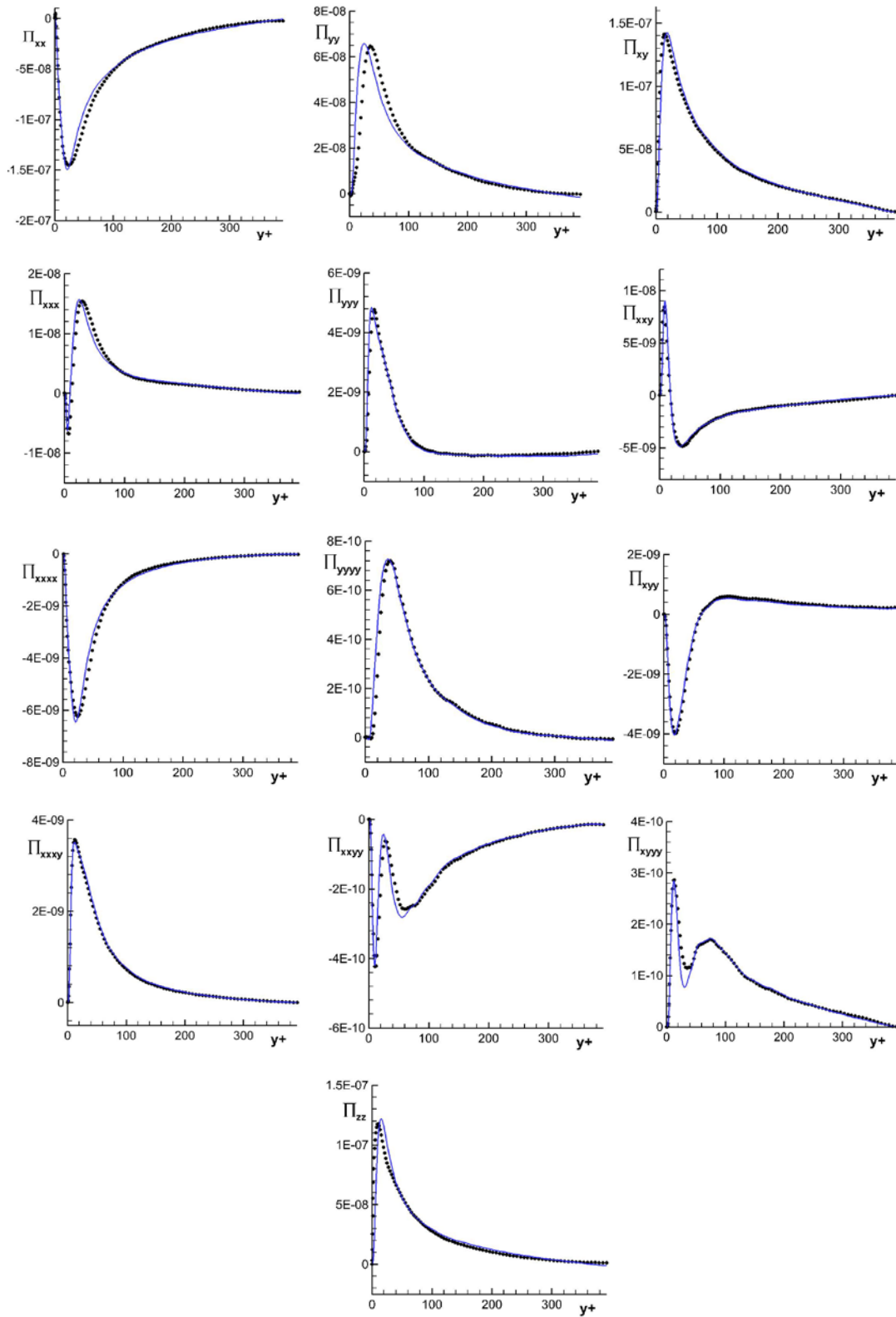


Figure 1. DNS and model profiles for the second-, third-, and fourth-order velocity/pressure-gradient correlations obtained using expressions (6) in a fully developed channel flow ($Re_\tau = 391.68$). Notations: $\bullet\bullet\bullet$ DNS .data [5]; — model profiles.