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Internal Solitary Waves in Two-Layer Fluids at Near-Critical Situation

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Abstract

A new model equation describing weakly nonlinear long internal waves at the interface between two thin layers of different density is derived for the specific relationships between the densities, layer thicknesses and surface tension between the layers. The equation derived and dubbed here the Gardner–Kawahara equation represents a natural generalisation of the well-known Korteweg–de Vries (KdV) equation containing the cubic nonlinear term as well as fifth-order dispersion term. Solitary wave solutions are investigated numerically and categorised in terms of two dimensionless parameters, the wave speed and fifth-order dispersion. The equation derived may be applicable to wave description in other media.

Introduction

The governing equation describing long internal waves of small amplitude at the interface in a two-layer fluid is, in general, the well-known KdV equation [1, 3]:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0 \qquad (1)$$

Here the coefficients are:

$$c^2=\frac{g(\rho_2-\rho_1)h_1h_2}{\rho_1h_2+\rho_2h_1},\quad \alpha=\frac{3c}{2h_1h_2}\frac{\rho_2h_1^2-\rho_1h_2^2}{\rho_1h_2+\rho_2h_1},$$

$$\beta=\frac{h_1h_2}{6}c\frac{\rho_1h_1+\rho_2h_2-3\sigma/c^2}{\rho_1h_2+\rho_2h_1}, \qquad (2)$$

where index 1 pertains to the upper layer, index 2 – to the lower layer (see figure 1), $\rho_{1,2}$ are densities of the layers, $h_{1,2}$ are thicknesses of the layers, and σ is the surface tension between the layers. For the sake of simplicity we assume here that the 'rigid lid' approximation is used to filter the surface mode [4].

However, at certain conditions equation (1) degenerates because some of its coefficients vanish. In particular, the generalisation is required when the density interface is located near the half-depth of the fluid. In this case the coefficient of quadratic nonlinearity α becomes anomalously small, and one should take into consideration the next order nonlinear term, the cubic term, to balance the dispersion effect [3]. The corresponding equation is known as the Gardner equation:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}-\alpha_1u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0 \qquad (3)$$

where the cubic nonlinear coefficient is:

$$\alpha_1=\frac{21}{8c}\left[\frac{8}{7}\left(\frac{c}{h_1h_2}\right)^2\frac{\rho_2h_1^3+\rho_1h_2^3}{\rho_1h_2+\rho_2h_1}-\left(\frac{2\alpha}{3}\right)^2\right] \qquad (4)$$

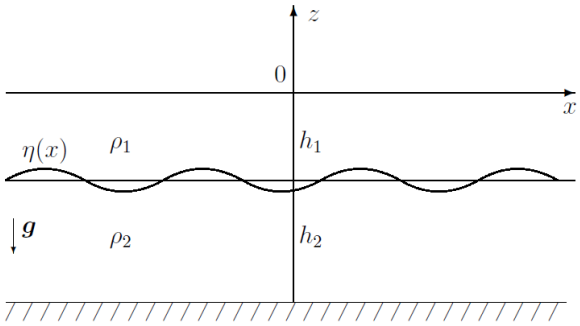

Figure 1. Sketch of internal waves at the interface between two fluid layers.

The KdV and Gardner equations are completely integrable [1]; they possess soliton solutions which attract a special interest due to their specific particle-like properties. In a two layer fluid the structure of Gardner solitons is very well studied.

There are also situations when the dispersion coefficient β vanishes. Such cases are known, for example for gravity-capillary waves with strong surface tension and magnetosonic waves in plasma [11]. In the near critical situation when the dispersive coefficient β becomes anomalously small, the next-order dispersion should be taken into consideration. This leads to the fifth-order KdV equation currently known as the Kawahara equation with the quadratic nonlinearity [12, 8, 7]:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\beta_1\frac{\partial^5 u}{\partial x^5}=0 \qquad (5)$$

Recently the similar equation was derived for internal waves for two-layer fluid [5] and it was obtained the expression for the coefficient β_1:

$$\frac{\beta_1}{cH^4}=\frac{b\left[4a\left(1+b^4\right)+19b\left(1+a^2b^2\right)+30ab^2\right]}{360\left(a+b\right)^2\left(1+b\right)^4}-$$

$$\frac{sb^2\left(1+a\right)}{32\left(1+b\right)^3\left(a+b\right)^2}\left[\frac{4}{3}\left(1+ab\right)+s\left(1+a\right)\left(1+b\right)\right] \qquad (6)$$

where $a = \rho_1/\rho_2$, $b = h_1/h_2$, $H = h_1 + h_2$, and $s = 2\sigma/[(\rho_1 + \rho_2)Hc^2]$.

It has been shown in [5] that in the case of internal waves in two-layer fluid with strong surface tension at the interface the double-critical situation is also possible when both the coefficients α and β become so small that the next order corrections with the coefficients α_1 and β_1 should be taken into consideration. As one can see from equation (2), both the coefficients α and β vanish simultaneously when $\rho_1 = \rho_2 b^2$ and $\sigma = c^2\rho_2 h_2(1 + b^3)/3$. In this case the coefficients α_1 and β_1 take simple forms:

$$\alpha_1 = \frac{3c}{h_1 h_2}, \quad \beta_1 = cH^4 \frac{1+b^5}{90(1+b)^5}. \qquad (7)$$

The coefficient β_1 is always positive and has the minimum value at $b = 1$; its plot is shown in figure 2.

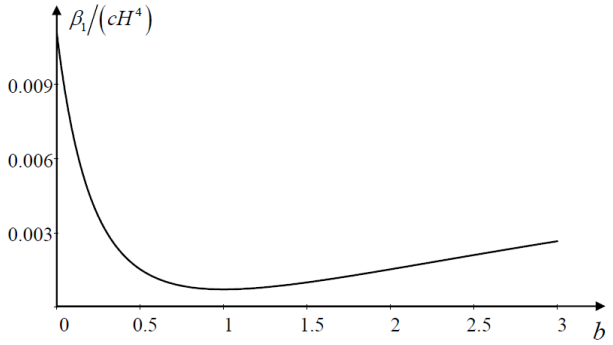

Figure 2. The dependence of the normalised dispersion coefficient $\beta_1/(cH^4)$ on the ratio of layer thicknesses $b = h_1/h_2$ at the double critical situation.

In the vicinity of the double critical situation the governing equation reads:

$$\frac{\partial u}{\partial t} + \left(c + \alpha u - \alpha_1 u^2\right)\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} + \beta_1\frac{\partial^5 u}{\partial x^5} = 0 \qquad (8)$$

We will call this equation the Gardner–Kawahara equation. In this paper we consider stationary solutions of equation (8) in the form of solitary waves (solitons). A family of such solutions are constructed numerically by means of the Petviashvili method [15, 14] and Yang–Lakoba method [16].

Dispersion relation and stationary solutions

Let us present equation (8) in the dimensionless form using change of variables:

$$\tau = \frac{\alpha^3 t}{\alpha_1\sqrt{\alpha_1\beta}}, \quad \xi = \frac{\alpha(x-ct)}{\sqrt{\alpha_1\beta}}, \quad \upsilon = \frac{\alpha_1}{\alpha}u. \qquad (9)$$

After that the main equation reads:

$$\frac{\partial \upsilon}{\partial \tau} + \upsilon\frac{\partial \upsilon}{\partial \xi} - \upsilon^2\frac{\partial \upsilon}{\partial \xi} + \frac{\partial^3 \upsilon}{\partial \xi^3} + B\frac{\partial^5 \upsilon}{\partial \xi^5} = 0 \qquad (10)$$

where $B = (\beta_1/\alpha_1)(\alpha/\beta)^2$. For waves of infinitesimal amplitude we obtain from equation (10) the following dispersion relation

$$\tilde{\omega} = -\kappa^3 + B\kappa^5, \quad V_{ph}(\kappa) \equiv \frac{\tilde{\omega}}{\kappa} = -\kappa^2 + B\kappa^4 \qquad (11)$$

The phase speed $V_{ph}(\kappa)$ is shown in figure 3.

In the case $B \le 0$ the phase speed is a monotonic function of κ, whereas at $B > 0$ it has a minimum, $V_{min} = -1/(4B)$ at the point $\kappa_c = (2B)^{-1/2}$. The concept of phase speed is very important in understanding the process of interaction of a moving source with waves. In particular, if the speed of a source is such that there is no resonance with any wave, i.e. there is no intersection of the dashed line in figure 3 with the dispersion curve (e.g., with lines 1 or 3), then the source does not loose energy for wave excitation. Otherwise, in the case of the resonance (see the intersection of dashed line with line 2), the source experiences energy losses for wave generation and, as a result, it experiences wave resistance. Without external compensation of energy losses such source cannot move stationary.

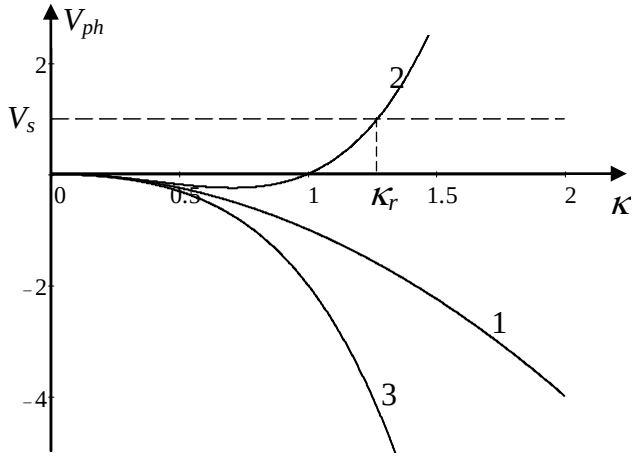

Figure 3. Phase speed for $B = 0$ (line 1), $B = 1$ (line 2), and $B = -1$ (line 3).

Consider now stationary solitary solutions to equation (10) depending on one variable $\zeta = \xi - V\tau$. Then, integrating equation (10) once we obtain:

$$B\frac{d^4\upsilon}{d\zeta^4} + \frac{d^2\upsilon}{d\zeta^2} - V\upsilon + \frac{\upsilon^2}{2} - \frac{\upsilon^3}{3} = 0. \qquad (12)$$

As follows from this equation the shape of a solitary wave is determined by two parameters, B and V. Considering asymptotic solution when $\upsilon \to 0$ at $\zeta \to \infty$, we can linearise this equation and seek for its solution in the form $\upsilon \sim e^{\mu\zeta}$. Then we obtain for μ

$$B\mu^4 + \mu^2 - V = 0 \qquad (13)$$

The roots to this bi-quadratic equation are:

$$\mu_{1,2}^2 = \frac{-1+\sqrt{1+4BV}}{2B}; \quad \mu_{3,4}^2 = \frac{-1-\sqrt{1+4BV}}{2B}. \qquad (14)$$

Assume first that B is negative, then for $V < 0$ we have $\sqrt{1+4BV} > 1$; therefore the roots $\mu_{1,2}$ are purely imaginary, and the roots $\mu_{3,4}$ are real. Solutions corresponding to purely imaginary roots are not decaying and can't represent solitary waves with the zero asymptotics. If $V = 0$, then $\mu_{1,2} = 0$ and again we have a solution with non-decaying asymptotics. If $0 < V < -1/(4B)$, then we have $\sqrt{1+4BV} > 0$, and all four roots μ are real. In this case the soliton solutions are possible with the exponential asymptotics at the infinity, $\upsilon \sim \exp(-|\mu_1\zeta|)$. And at last, if $V > -1/(4B)$, then $\sqrt{1+4BV}$ is complex; all roots are complex-conjugate in pairs $\mu_{1,2} = \pm(p_1 \pm iq_1)$, $\mu_{3,4} = \pm(p_2 \pm iq_2)$. Due to the presence of the real parts of the roots $p_{1,2}$, the soliton solutions are also possible with the oscillatory asymptotics. The decay rate of a solitary wave in the far filed is determined by the root with the smallest value of $|p_{1,2}|$.

Assume now that B is positive, then it follows from the similar analysis as above that solitary waves with the oscillatory

asymptotics are possible only when $V < V_{min} \equiv -1/(4B)$. In the particular case of $B = 0$, equation (13) has two real roots $\mu_{1,2} = \pm\sqrt{V}$ corresponding to soliton solutions, provided that $V > 0$.
These findings can be summarised with the help of a schematic diagram shown in figure 4. It should be noticed that the analysis of roots only predicts *possible* asymptotics of solitons provided that they exist, but it does not guarantee their existence. In particular, if $B = 0$, then soliton solutions with monotonically decaying exponential asymptotics could exist for any $V > 0$ (see case b) in the diagram), but in fact they exist only for $0 \le V \le 1/6$ (see the exact solution (15) of the Gardner equation below).

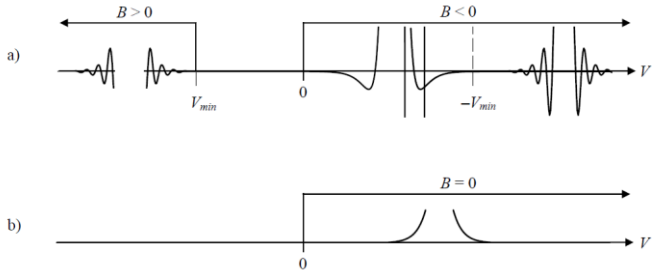

Figure 4. Schematic structure of solitary wave asymptotics for equation (12) with different signs of B (frame a). If $B = 0$ only the solitons with monotonic exponential tails are possible as shown in frame b) – see exact solution equation (15) below.

Soliton solutions to the Gardner–Kawahara equation

Consider first the limiting case of $B = 0$ when equation (12) reduces to the completely integrable Gardner equation. It has a family of soliton solutions which is determined by only one parameter V varying in the interval $0 \le V \le 1/6$ (a soliton solution does not exist beyond this interval):

$$\upsilon(\zeta) = \frac{\sqrt{6V}}{2}\left[\tanh\left(X + \phi\right) - \tanh\left(X - \phi\right)\right], \quad (15)$$

where $X = \zeta\sqrt{V}/2$, $\phi(V) = (1/4)\ln\{[1 + (6V)^{1/2}]/[1 + (6V)^{1/2}]\}$, and soliton amplitude $A = 1 - (1 - 6V)^{1/2}$. The family of solitons varies from KdV-type bell-shaped solitons, when $V \to 0$, to table-top solitons, when $V \to 1/6$ [3]; the typical solitons are shown in figure 5 for three values of V.

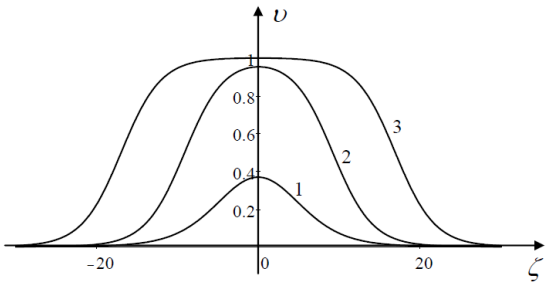

Figure 5. Solitary wave shapes as per equation (15) for three values of V. Line 1 pertains to $V = 0.1$ (quasi-KdV case), line 2 pertains to $V = 0.1663$ ('fat soliton'), and line 3 pertains to $V = 0.166666$ (table-top soliton).

If $B \neq 0$, then exact analytical solutions of equation (12) are not known. However they can be constructed numerically using, for example, a modified Petviashvili method [15, 14] or even more effective Yang–Lakoba method [16].
As the test case we have considered the Gardner equation (12) with $B = 0$. For relatively small $V \le 0.1663$ we obtained numerical solutions in a complete agreement with the analytical solution (15). But for larger values of V we failed to obtain any solution by means of the modified Petviashvili method as the iteration procedure did not converge to something. In the meantime, the application of the Yang–Lakoba method [16] enabled us to construct soliton solutions in the complete agreement with the analytical solution (15) for any positive value of $V < 1/6$. Thus, we have confirmed that the Yang–Lakoba method is even more efficient and converges faster than the Petviashvili method.
In the case of $B > 0$ we obtained various shapes of solitary waves by the Petviashvili method for different values of V; they are illustrated by figure 6 for $B = 1$. For other positive values of B the soliton structures are qualitatively the same as shown in this figure, but solitons become narrower when $B \to 0$ and wider when $B \to \infty$. Their asymptotic behaviour is in agreement with the prediction shown in figure 4 for $B > 0$; the oscillatory tails of solitons are clearly seen in this figure.

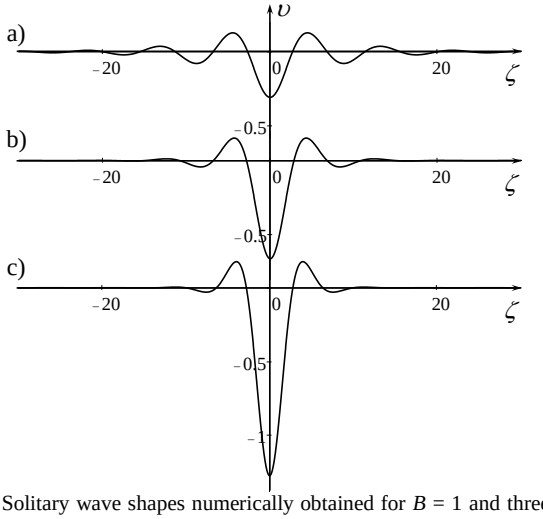

Figure 6. Solitary wave shapes numerically obtained for $B = 1$ and three values of V. (Case a) $V = -0.3$, (case b) $V = -0.5$, (case c) $V = -1.0$.

It is worth noting two features of these solitons. Firstly, at small amplitudes, when $V \to (V_{min})$., solitons reduce to stationary moving wavetrains. Such wavetrains can be described by the higher-order non-linear Schrödinger equation; similar solutions in the form of envelope solitons have been earlier obtained for the Ostrovsky equation [13, 9].
Secondly, due to oscillatory character of soliton tails, solitons can form the bound states – stationary propagating bi-solitons and even more complicated multi-solitons [6, 7, 13]. We do not consider such structures in details here, but present only one example in figure 7. In frame a) one can see a single soliton with oscillating tails, whereas in frame b) the bi-soliton is depicted for the same parameters $B = 1$ and $V = -0.5$.

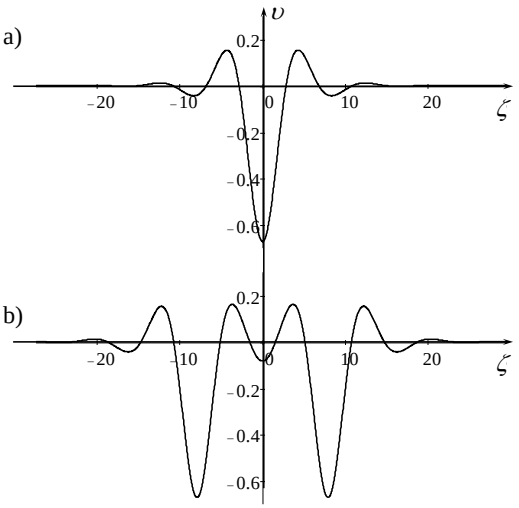

Figure 7. Examples of single soliton and bi-soliton representing a family of stationary multi-soliton solutions of the GK equation (12).

For the sake of completeness, we have investigated a structure of soliton solutions for the negative parameter B too, although in the

context of interfacial waves in two-layer fluid this parameter is always positive. With the help of Petviashvili and Yang–Lakoba numerical methods we have obtained a family of soliton solutions only in the finite range of the parameter V: $0 < V \le V_{cr}$, where V_{cr} depends on B, but only slightly differs from 1/6. We did not investigate in details the dependence of soliton speed on its amplitude for different values of B, this can be a matter of a separate study. In figure 8 we present several soliton solutions constructed numerically for the fixed amplitude A = 0.998 and different B.

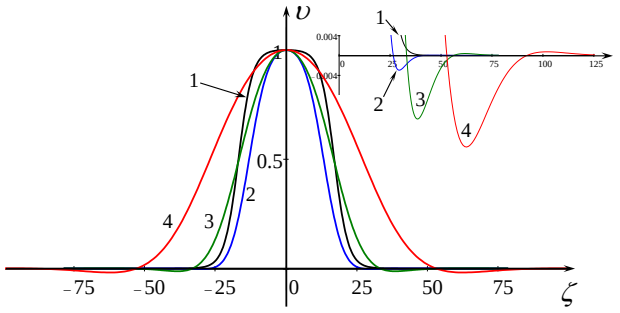

Figure 8. Solitary wave shapes numerically obtained for the fixed soliton amplitude A = 0.998 and four values of B: (line 1) B = 0, (line 2) B = –10, (line 3) B = –100, (line 4) B = –1000.

As one can see from this figure, the greater the B, the wider the soliton. Again, in accordance with the prediction, soliton asymptotics changes from the exponential aperiodic, when $V < -1/(4B)$, to oscillatory decaying, when $V > -1/(4B)$ (see figure 4). However, the oscillations are so small and they decay so fast that are hardly visible in soliton profile. In the insertion one can see the magnified portion of corresponding soliton tails.

Due to oscillatory character of soliton tails the bounded bi-soliton and multi-soliton solutions may exist in this case too. Moreover, even more complicated infinite chains of bounded solitons are possible. Such chains may be both regular and irregular representing quasi-random sequences of bounded solitons [7].

Conclusion

Thus, in this paper we have shown that in the study of interfacial waves between two immiscible fluids there are such situations, when the double critical conditions can occur, i.e. when both the coefficients of quadratic nonlinearity and third-order dispersion vanish simultaneously. In the near-critical situation the basic governing equation is the Gardner–Kawahara equation (8). Let us make an estimate of physical applicability of the Gardner–Kawahara equation to the real case of two-layer system consisting of kerosene in the upper layer and water in the lower layer. Take the following parameters for the kerosene and water at 20°C: the density of kerosene is ρ_1 = 0.82 g/cm^3, the density of water is ρ_2 = 0.998 g/cm^3, and the surface tension at the interface between them is σ = 48 dyn/cm. Then the double critical condition occurs, if the layer thicknesses are h_1 = 0.82 cm and h_2 = 0.9 cm with the total fluid depth $H = h_1 + h_2$ = 1.72 cm. If we choose h_1 = 0.62 cm and h_2 = 1.1 cm with the same total depth, we will have a near critical condition with $h_1/h_2 \approx 0.56$. In this case the coefficient β_1 is positive and close to its minimum value (see figure 2); the dimensionless coefficient B in equations (10), (12) is also positive, and equation (12) possesses soliton solutions of negative polarity with the oscillatory tails as shown in figure 6. The shapes of such solitons may vary from quasi-sinusoidal wavetrains, when the amplitude goes to zero, up to narrow pulses, when the amplitude increases. The negative speeds of these solitons (see the diagram in figure 4) imply that they travel with the speeds less than the speed of long linear waves c in the immovable coordinate frame [see equations (1) and (2)].

Apparently, the similar Gardner–Kawahara equation can be applicable for the description of other types of waves in continuous media, for example, in plasma physics. The equation can be further generalised to take into account weak medium rotation in the spirit of the Ostrovsky equation [3, 10] and weak wave diffraction in the spirit of the Kadomtsev–Petviashvili equation [15]. In the latter case two-dimensional lump solitons with oscillatory tails can be possible. Such solutions were constructed numerically for the two-dimensional version of the Kawahara equation (3) in [2].

Acknowledgments

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