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On the Stability of Time–Periodic Pipe Flow

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Adumbrate in a century of determined scientific investigation is the significance of pipe-flow research. The transition to turbulence of fluid flows though these rigid, circular geometries is of interest in many and often diverse branches of science and engineering. Despite this context, very little is know of the mechanisms and instabilities responsible for moving the system away from the laminar basin.

It is known that laminar Hagen–Poiseuille flow is linearly stable to general perturbations at finite Reynolds numbers, and that single-harmonic periodic flow is stable to axisymmetric perturbations. However, the stability of non-axisymmetric time-periodic flows is yet to be tested by conventional numerical means. We extend the analysis to demonstrate that the pure oscillatory flow is also stable to general perturbations. We explain how this implies that all laminar steady and periodic circular pipe flows of this type are linearly stable. The least stable modes identified are axially invariant. The results of the stability study are discussed in comparison to Direct Numerical Simulation (DNS).

Introduction

As is well known, steady laminar flow in a circular tube (Hagen–Poiseuille flow) is linearly stable to general infintesimal disturbances for all Reynolds numbers yet studied (e.g. (4)) but is observed to become turbulent at bulk flow Reynolds numbers of order $Re = UD/\nu = 2000$–3000 in moderately careful experiments; where U is the bulk flow speed, D is the pipe diameter and ν the kinematic viscosity. Under careful experimental conditions this transition point can be delayed well beyond these values. There is still debate about the precise mechanism that leads to transition.

Here we are concerned with the stability and transitional mechanisms of time-periodic flows though pipes, which may be either oscillatory (zero time-average bulk flow) or pulsatile (non-zero time average bulk flow). Oscillatory and pulsatile incompressible flows in a straight rigid circular tube are canonical phenomena of classical fluid mechanics. In addition they serve as models of a variety of flows of engineering and physiological application, for example peristaltic pumping and arterial flows. For these cases it is useful to define the peak bulk-flow-speed ($\bar{u}_p$) as:

$$\bar{u}_p = \max_{0<t\leq T} \bar{u}(t)\,, \tag{1}$$

where T is the period, and $\bar{u}$ the area-average or bulk-flow-speed:

$$\bar{u}(t) = \frac{8}{D^2}\int_0^{D/2} u(r,t)\, r dr. \tag{2}$$

Previous work by (8) on axisymmetric perturbations of single-harmonic oscillatory flows found such regimes linearly stable. The case for general perturbations is a subset of our extension to the literature. Experiments in these flows (e.g. (3)) show that transition to turbulence can occur, often in the form of bursts during each oscillation. In the case of pulsatile pipe flow (when an oscillation is superimposed on a steady mean flow), experiments, e.g. by (6), also demonstrate the presence of burst-type transition. However in this case, there is presently no published study of linear stability, a deficiency that our present work aims to remedy.

The laminar base-flows for these problems are obtained from the analytical solution of the Navier–Stokes equations (5) in cylindrical coordinates. For a prescribed bulk-area-average flow rate (2) – assumed T-periodic – the corresponding radial velocity profile conforms to the Fourier–Bessel function derived by (5) and is couched here in terms of the Womersley number (Wo):

$$u_n(r,t) = \mathbf{R}\left[\frac{K_n iT}{\rho 2\pi n}\left(\frac{J_0\left(i^{3/2} Wo 2r/D\right)}{J_0\left(i^{3/2} Wo\right)} - 1\right)\exp 2\pi int/T\right]. \tag{3}$$

Here, n is a frequency harmonic, J_0 is a complex Bessel function and K_n is an associated complex axial pressure gradient. In the limit as T grows without bound, this analytical solution asymptotes to the standard parabolic Hagen–Poiseuille solution. It is important to note that $Wo = \sqrt{2}\cdot R/\delta = R\cdot\sqrt{\omega/\nu}$ is the Womerley number – a non-dimensional frequency parameter, based on ω; the oscillatory frequency. Associated is the $\delta = \sqrt{2\nu/\omega}$ length scale, known as the Stokes layer thickness over a flat-plate in oscillatory flow. This is used to scale wall-units, denoted by $^+$ ($x^+ = x\delta/\nu$).

Without loss of generality we can ignor the pressure gradient K_n as a parameter and adjust the phases and amplitudes of the solutions to (3) such that at each temporal harmonic n, we have

$$\bar{u}(t) = \sum_n \left[A_n \cdot cos(n\omega\cdot t/T) + B_n \cdot sin(n\omega\cdot t/T)\right]. \tag{4}$$

The $n = 0$ case corresponds to the standard Hagen–Poiseuille solution $u(r) = A_0\left[1-(r/R)^2\right]$, and as stated above, is also a solution to (3). In this – the steady flow case – the only parameter is the Reynolds number, Re. Alternatively, for the time-periodic cases the flows have two dimensionless parameters that describe the pulse period and some measure of the flow speed. Taking $\bar{u}_p$ as a velocity scale and diameter D as a length scale, the time scale is $D/\bar{u}_p$. This leads to a choice of the two dimensionless parameters; a Reynolds number and reduced velocity, respectivly:

$$Re = \frac{\bar{u}_p D}{\nu} \quad \text{and} \quad U_{red} = \frac{\bar{u}_p T}{D}.$$

The mean flow velocity scale can again appear though the reduced velocity. This pairing is a sensible choice for the oscillatory cases in that the Womersley number appears in the analytical solution for the base flows. We note that the oscillatory components of the base flow have (via eq. 3) velocity profiles that are only a function of Wo, r/R, and t. The reduced velocity is then a premultiplying kinematic factor that describes how far the bulk flow oscillates along the pipe, expressed in pipe diameters, but does not alter the velocity profile. For oscillatory flows

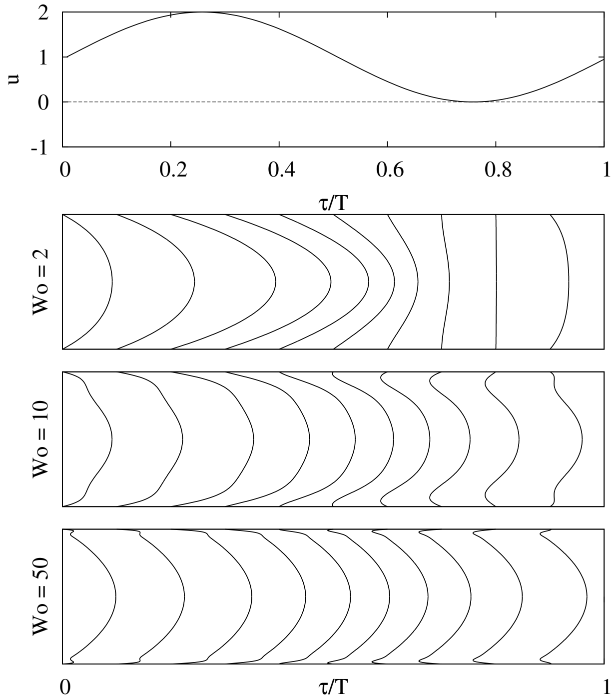

Figure 1: Base flows of $\bar{u}(r,t)$ for period points $t/T = 0 \to 1$. As $Wo > 1$ the base-flow becomes more plug-like.

we nonetheless would expect *a priori* that their stability could be a function of the two dimensionless flow parameters, *Wo* and reduced-velocity, as well as axial (α) and azimuthal wavenumbers (k).

Since all the flows have the same boundary conditions, we can consider their linear stabilty on a term–by–term bases, one term for each termporal Fourier harmonic. Again exploiting linearity, we can deal with general spatial perturbations at each temporal period as a linear sum of axial and azimuthal Fourier modes, with wave numbers $\alpha = 2\pi D/L_x$ and k respecitively. One implication is that we do not here need to examine the linear stabiilty of the steady flow, since that has been comprehensivly dealt wtih in previous works — it suffices to examine the stability of the oscillatory components, and these can be delt with one temporal harmonic at a time.

To orientate the reader, we present in Figure 1 the radial profiles of axial velocity for ten phase-points in the base flow cycle. These profiles correspond to the modulation of bulk flow speed in t/T (top of Fig. 1). With increasing *Wo*, the velocity profile becomes more like plug flow but with small overshoots near the pipe wall.

Methodology

Stability Analysis

The stability analysis problem is solved in primitive variables. Starting from the incompressible Navier–Stokes equations,

$$\partial_t u = -u \cdot \nabla u - \nabla p + \nu \nabla^2 u, \quad \nabla \cdot u = 0, \tag{5}$$

where p is the kenematic or modified pressure. It is prorpsed that $u = U + u'$ where U is the base flow whos stability is examined and u' is an infinitesimal perturbation. Upon substitution and retaining terms linear in u', the linearized Navier–Stokes equations are obtained:

$$\partial u' = -u' \cdot \nabla U - U \cdot \nabla u' - \nabla p' + \nu \nabla^2 u'. \tag{6}$$

We note that in the present problem, the base flow is T-periodic, i.e. $U(t+T) = U(t)$. Because in incompressible flows the pressure is not an independant variable, and all terms are linear in u', we can write this evolution equation in symbolic form,

$$\partial_t u' = L\left(u'\right), \tag{7}$$

where L is a linear operator with T-periodic coefficients through the influence of the base flow. Correspondingly the stability of this equation is a linear temporal Floquet problem. Writing the state evolution of u' over one period as

$$u'(t+T) = A(T)\, u'(t), \tag{8}$$

where $A(T)$ is the system monomodry matrix, we obtain a Floquet eigenproblem:

$$A(T)\, u_j''(t) = \mu_j u_j''(t). \tag{9}$$

Here $u_j''(t)$ are phase-specific Floquet modes and μ_j are Floquet multipliers (which generally occur in complex-conjugatepairs). Stability of the problem is assessed from the Floquet multipliers: unstable modes have multipliers that lie outside the unit circle in the complex plane (i.e. $|\mu| > 1$), while stable modes lie inside (i.e. $|\mu| < 1$).

We use a time-stepping based methodology outlined in (7) and given detailed explanation in (1) in order to solve the Floquet eigenproblem. A key point about the approach is that a system monodromy matrix $A(T)$ is not explicitly constructed; rather, a Krylov method is used that is based on repeated application of the state transition operator (7) whos action is obtained by integrating the linearised Navier–Stokes equations forward in time over interval T. By varying the Krylov dimension and ensuring sufficient resolution we are typically able to resolve a moderate number (e.g. ten) of the leading (least stable) Floquet modes.

Numerical methods

Spatial discretization and time integration is handled using a cylindrical coordinate spectral element method with mixed explicit/implicit time stepping, as outlined in (2). The domain is discretized into spectral elements in the meridional semiplane that runs from the pipe axis to the outer radius in the radial direction and a finite length of pipe L_z in the axial direction. For all cases investigated the mean y^+ was 0.3626 with a $\sigma = 0.1970$.

Fourier modal stucture is assumed in the azimuthal direction with integer wavenumbers k, and as a result of linearization, each azimuthal mode can be dealt with independently. In the axial direction we use real wavenumbers $\alpha = 2/L_z$. Because of the approach taken to spatial discretization in the axial direction, the Floquet eigensolution for an domain length can contain modes for both $\alpha = 0$ (i.e. modes that are axially invariant) and $\alpha = m2/L_z$ (where m is an integer). Typically, there are a number of multipliers for $\alpha = 0$ that are larger in magnitude than the first axially varianet mode and we compute sufficient modes to be assured that we obtain the leading mode for $\alpha = 2\pi D/L_z$ as well.

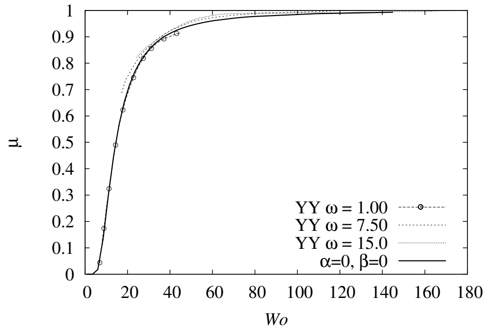

Figure 2: Floquet multiplier data derived from (8) (YY in graph), for the dominant axially invariant axisymmetric modes. Compilation of comparable results obtained from present computations shown as a solid line ($\alpha = 0$, $k = 0$).

The same spatial and temporal structure is implemented for the flow direct numerical simualtion (DNS). However the azimuthal resolution in Fourier space was compressed to $k = 0 \rightarrow 32$ (64 planes). Global body forcing to modulate the oscillatory terms of (4) between $\bar{u}_p = \pm 1$ was based on the laminar solution to (3).

Results

We first examine stability to axisymmetric perturbations ($k = 0$), as dealt with previuosly by (8), who found that axially invariant modes ($\alpha = 0$) were the least stable. We have re-interpreted their dimensionless groups as *Wo* and U_{red} and presented their results, along with ours, for $\alpha = 0$, $k = 0$ as shown in Figure 2. In all cases, the flow is stable ($\mu < 1$), but only marginally so at large *Wo*.

Our data compares well with those from (8) (the slight discrepancies seen are attributable noise in our digitization of their figures). Of note is the collapse of three figures from (8) onto a single curve for our choice of dimensionless groups. We note that this collapse is not seen in their data for non-axially invariant modes, i.e. $\alpha > 0$. Our results, to be discussed below, also show that for $\alpha > 0$, there are (as expected *a priori*) again two dimesionless groups. This anticipation stems from the U_{red} parameter pertaining only to Floquet modes having axial structure. Axially invariant modes are then dependent only on the *Wo* control.

The stability of general perturbations ($k > 0$) was investigated for a series of *Wo* at varying levels of three-dimensionality. These were compared to the axisymmetric case ($k = 0$) for the axial wavenumber $\alpha = 0$. In Figure 3 only the dominant (least stable) Floquet multiplier (μ) is presented for each case.

The axialy invariant case is of particular interest as it allows the multiple curves associated with the U_{red} kinetic parameter to be condensed, and the results interpreted soly on the basis of the Womerley number. From Figure 4 we can summize that $\alpha = 0$ is the leading branch for the $k = 0$ leading mode. Hence, confirming the results of (8). The behavior of sub-dominant modes – presented in Figure 5 – confirms the Floquet stability of the system.

Encapsulated in 4 by linear superposition are all periodic pipe flow profiles. By noting the linearity of the Floquet decomposition, and that of the boundary conditions, the system as a whole is assessed for stability from its constituent harmonics – all of which are linearly stable to infantesimal perturbations. Hence,

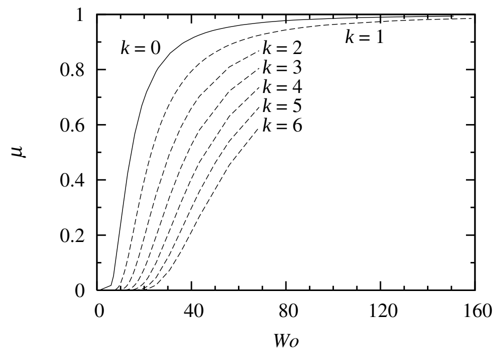

Figure 3: Dominant Floquet mulitpliers for axially invariant ($\alpha = 0$) modes. These were calculated for a range of *Wo* and azimuthal wave numbers (k).

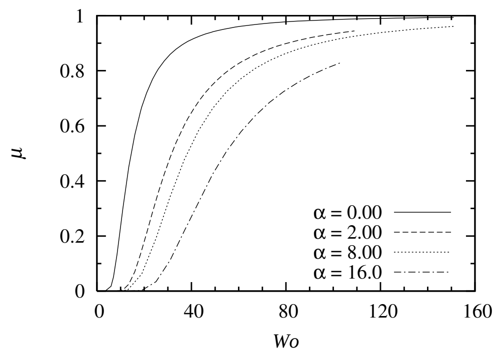

Figure 4: Floquet mulipliers for the $k = 0$ case over $\alpha \geq 0$.

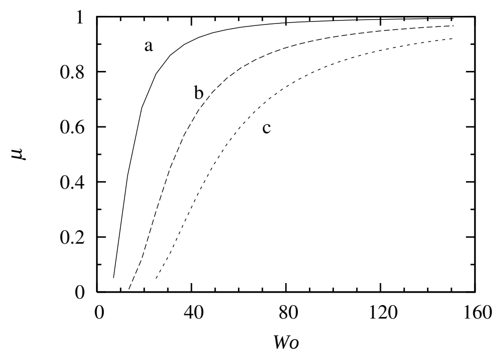

Figure 5: The leading three (a,b,c) Floquet multipliers for for $k = 0$, $\alpha = 0$

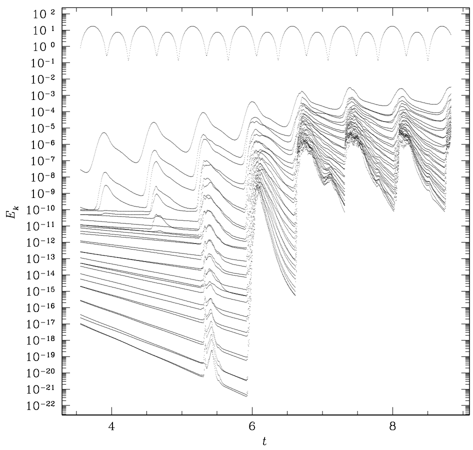

Figure 6: Modal energies (E_k) for a history of dimensional t ($Wo = 66$, $U_{red} = 6.83$). Initially all modes decrease monatonically from $k = 0$ in E_k. The first mode to recieve amplification is $k = 1$, which then picks-up $k = 2$, and so on infinitem leading to complete transition.

all time-periodic flows though pipes are linearly stable.

Direct Numerical Simulation

The relative influence of the $k = 0$ wave number was investigated by direct numerical simulation of both oscillatory and pulsatile pressure-driven flows. Base flows of type (3) were perturbated with $\sum E_{k=0\to31} = 10^{-4}$, where E_k is the energy in mode k as per the standard:

$$E_k = \frac{1}{2A}\int_{\Omega} \mathbf{u}_k^* \cdot \mathbf{u}_k \, r \, d\Omega, \qquad (10)$$

here Ω is the meshed spectral-element plane (of which A is the area) and $\mathbf{u}$ is the velocity field with $\mathbf{u}^*$ the complex-conjugate. For brevity we present only a single case.

The growth of modes in the periodic flow is dominated by, and coupled to the axisymmetric mode. Subsequent modes experience self-similar fluctuations to $k = 1$. Modes $k = 1 \to 4$ experience significant growth, not predicted by the linear theory.

The peaking of the modal energy is followed by a sink towards a laminar recovery position. For the transitional period ($t = 4 \to 6$) this means a return to a laminar profile from turbulent patches; Figure 7. For high Wo there is very little phase-lag between the recovery of the near-wall regions and that of the centre-line. Perturbations near the wall have been found (3) to be several orders of magnitude smaller when compared to the centre-line. Such phase-locked disturbances are characteristic of periodic flows.

Conclusion

The present study extends (and confirms) the work of (8) to general perturbations. We have found the least-stable Floquet mode to be axisymmetric and axially invariant. Additionally, by use of the Womerley number parameter – and noting the lineariarity of both the operator and boundary conditions – we have determined that all time-periodic pipe flows are vanishingly stable. This result, coupled with the known stability of Hagen–Poiseuille flow, renders the comlpete family of pipe-flows linearly stable.

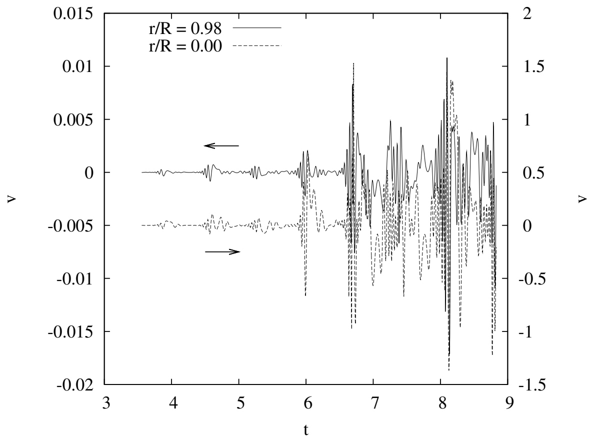

Figure 7: Fluid velocity in the radial (v) direction over the transitional time; taken at two locations in the pipe – near the wall ($r/R = 0.98$), and at the pipe centre-line ($r/R = 0$). At $Wo = 66$ there is very little phase-lag for $r/R = 0.98 \to 0$. Phase-locked turbulent patches are a signature of transitional periodic flows.

Further work in DNS for time-periodic pipe flow has shown that the azimuthal wave modes are highly coupled. The largest, by energy, is $k = 0$. We also demonstrate the active role that $k = 1 \to 4$ take in the transitional stages of flow. Further work in transient growth is required to support a by-pass transition conclusion.

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