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Evolution of zero pressure gradient turbulent boundary layers

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Abstract

The streamwise evolution of a turbulent boundary layer (TBL) developing under constant pressure on a smooth wall is considered. The closure problem is described for the zero pressure gradient (ZPG) flow where the only assumptions made are the use of classical similarity laws, such as Prandtl's law of the wall and Coles'[1, 2] law of the wake, together with the mean continuity and mean momentum differential and integral equations. The important parameters are identified and the problem is reduced to one semi-empirical input with the assumption that the Reynolds shear stress can be described by a two-parameter family. Good agreement is shown with the experimental data.

Introduction

Here we compute the evolution of turbulent boundary layers developing in a zero-pressure gradient. Unlike channel or pipe flows where there is a minimum development length for the flow to be regarded as *fully developed*[1], a boundary layer is continuously evolving and can therefore never be *fully developed*. Comparisons of previous experimental data sets (as discussed in the review of Marusic *et al* [4]) have shown significant differences between different studies even though local parameters, such as R_θ (Here, R_θ is the Reynolds number based on free stream velocity (U_1) and momentum thickness (θ).) is matched. A likely cause for these disparities are the different evolution conditions for each boundary layer. Nagib *et al* [8]) have discussed this issue in the context of a *well-behaved* ZPG boundary layer, but this again relates the specific initial conditions each experimental boundary layer evolves from. In this paper we will consider the issue of evolution and attempt to develop a framework for computing the evolution of ZPG boundary layers starting from arbitrary initial conditions.

Perry *et al* [9], initially developed a preliminary mathematical framework for computing the evolution of turbulent boundary layers. It was found that there are 4 parameters that control the streamwise evolution of the boundary layer and the Reynolds shear stress distribution namely S, Π, β and ζ. Here, $S = U_1/U_\tau$ where U_1 is the local free stream velocity and U_τ is the friction velocity. Π is the wake parameter, $\beta = (\delta^*/\tau_0)(\mathrm{dp/dx})$ is the Clauser pressure gradient parameter where δ^* is the displacement thickness, p is the free stream static pressure, τ_0 is the wall shear stress, x is the stream wise distance and the non equilibrium parameter, $\zeta = S\delta d\Pi/dx$ where δ is the boundary layer thickness. The definitions of the law of the wall and law of the wake can be used to formulate a mean velocity defect law of the form,

$$\frac{U_1 - U}{U_\tau} = F[\eta, \Pi]. \tag{1}$$

Perry *et al* [9] had shown that integration of the streamwise momentum integral equation of the form,

$$\frac{d\theta}{dx} + \frac{(H+2)\theta}{U_1}\frac{dU_1}{dx} = \frac{C_f}{2} \tag{2}$$

(Here, the shape factor, $H = \delta^*/\theta$. Where, δ^* and θ are the displacement thickness and momentum thickness respectively. C_f is the skin friction co-efficient) coupled with equation (1) leads to an expression for the total shear stress of the following form,

$$\frac{\tau}{\tau_0} = f_1[\eta,\Pi,S] + g_1[\eta,\Pi,S]\zeta + g_2[\eta,\Pi,S]\beta \tag{3}$$

where $\eta = z/\delta$ and z is the wall-normal distance. Here f_1, g_1 and g_2 are known analytical functions where their precise form depends on the wall-wake formulation of Jones [3],

$$\frac{U}{U_\tau} = \frac{1}{\kappa}\log\left[\frac{zU_\tau}{\nu}\right] + A - \frac{1}{3\kappa}\eta^3 + \frac{\Pi}{\kappa}2\eta^2(3-2\eta) \tag{4}$$

Here, $\kappa = 0.41$ is the Karman constant and A is the universal smooth wall constant, taken here to be 5.0. There are many wall-wake formulations available in the literature, for example, see [6].

Evolution and Closure Equations

Calculating the streamwise evolution of the ZPG turbulent boundary layer is possible by solving a coupled set of ODE's which come from the momentum integral equation, law of the wall, law of the wake and the definitions of ζ and β,

$$\frac{dS}{dR_x} = \frac{\chi[R_x,K]R[S,\Pi,\zeta,\beta]}{SE[\Pi]exp[\kappa S]} \tag{5}$$

$$\frac{d\Pi}{dR_x} = \frac{\zeta\chi[R_x,K]}{S^2E[\Pi]exp[\kappa S]} \tag{6}$$

Here,

$$\begin{aligned}
R &= \frac{S}{\kappa S^2C_1 - \kappa SC_2 + C_2} + \frac{\beta(2SC_1 - C_2)}{C_1(\kappa S^2C_1 - \kappa SC_2 + C_2)} \\
&+ \frac{\zeta\left(\frac{dC_2}{d\Pi} - S\frac{dC_2}{d\Pi} - 2(C_2 - SC_1)\right)}{\kappa SC_1 - \kappa SC_2 + C_2} \\
C_1 &= \int_0^1((U_1 - U)/U_\tau)d\eta \\
C_2 &= \int_0^1((U_1 - U)/U_\tau)^2 d\eta \\
R_x &= xU_0/\nu \\
\chi[R_x,K] &= U_1(x)/U_0 \\
E[\Pi] &= exp[-\kappa A - 2\Pi + 1/3] \\
\beta &= -C_1S^2E[\Pi]exp[\kappa S]K
\end{aligned} \tag{7}$$

where U_0 is the reference free stream velocity (free stream velocity on top of the trip, i.e, $U_0 = U_1[0]$), R_x is the Reynolds number at different stream wise stations.

[1]The definition of *fully developed* requires that all mean flow quantities and all higher order turbulence quantities should become independent of streamwise location.

Evolution equations were restricted in the initial work of Perry *et al* [9] and the only problem that could be solved was the so called quasi equilibrium[2] flow cases where it could be assumed that ζ was sufficiently small to neglect its effect even though Π is permitted to vary with x. Thus the problem reduces to considering the relation,

$$C[\Pi,\beta,S]=0 \tag{8}$$

where for a given Π it is assumed that the velocity defect distribution is fixed and the shear stress distribution is fixed (approximately). Hence from data, if we know β at a given S for a fixed Π (i.e. for one experimental data point), then for this fixed Π we can find β versus S for all S using equation (3) to ensure that τ/τ_0 profiles are matched (approximately for all S). It is found that for S sufficiently large $\beta=\beta_a$ (the asymptotic value of β) and C is no longer a function of S. If this procedure is repeated for different values of Π, a one-to-one relationship between β_a and Π can be found which is based on experiments. This formulation is consistent with the universal relation for eddy viscosity ε, i.e. $\varepsilon/(\delta_c U_\tau)=\phi[\eta,\Pi]$. Unfortunately such formulations are known to break down in non-equilibrium flows, i.e. flows with significant ζ contribution - see Marusic & Perry [5] for an example.

In the case of zero pressure gradient boundary layers,

$$\begin{aligned} U_0 &= U_1\\ \chi[R_x,K] &= 1\\ K &= \frac{\nu}{U_1^2}\frac{dU_1}{dx} \end{aligned} \tag{9}$$

and the acceleration parameter, $K=0$, because the free stream velocity (U_1) is invariant with respect to x (i.e. $dU_1/dx=0$). Therefore, in summary, the streamwise evolution of ZPG flow can be computed using equations (5), (6) and (7). Here we have to solve three equations to find four unknowns namely S, Π, ζ and β. This is not possible unless we have a fourth equation comprising all the unknown parameters and therefore the fourth (i.e., closure) equation has to be derived empirically, which is of the following form,

$$F[\Pi,S,\beta,\zeta]=0. \tag{10}$$

For ZPG turbulent boundary layer, Clauser's pressure gradient parameter, $\beta=0$ which reduces (10) to the following form,

$$F[\Pi,S,\zeta]=0. \tag{11}$$

Now, a two parameter family of shear stress profiles of the form,

$$\frac{\tau}{\tau_0}=f[\eta,\Pi,\zeta_a] \tag{12}$$

is suggested and when used in conjunction with (3) some information can be obtained regarding formulation (10) as follows. Consider the (S,ζ)-plane. If such a plane contains an experimental data point, then S, Π and ζ are known for that data point and so also is τ/τ_0 versus η from (3). Trace out a curve for increasing S of fixed shear stress profile shape on the (S,ζ)-plane. By taking $S\to\infty$ we obtain asymptotic values of ζ (i.e., ζ_a). Going to $S\to\infty$ is simply a convenient curve-fitting procedure and cannot be approached experimentally.

This process of keeping the profile shape fixed will be referred to as *profile matching* which was given by Perry *et al* [10]. i.e.,

$$\frac{\partial}{\partial\zeta}\left[\int_0^1\left[\left(\frac{\tau}{\tau_0}\right)-\left(\frac{\tau}{\tau_0}\right)_D\right]^2 d\eta\right]=0 \tag{13}$$

where τ/τ_0 and $(\tau/\tau_0)_D$ are the shear stress distribution at any point on the (S,ζ)-plane for a fixed Π and at a known datum point respectively. Taking (13) to $S\to\infty$ means that $\partial/\partial\zeta\equiv\partial/\partial\zeta_a$ as the derivative and we can show generally that

$$A[\Pi,S]+B[\Pi,S]\zeta=C[\Pi]\zeta_a \tag{14}$$

where A, B and C are known analytical functions. Their precise form depends on the law of wall-wake formulation (4). If this process is repeated for different Πs then we obtain a relationship between Π and ζ_a and we thus have a known function ψ,

$$\psi[\Pi,\zeta_a]=F(\Pi,S\to\infty,\zeta_a)=0 \tag{15}$$

which is the asymptotic closure equation.

Experimental Setup

In order to evaluate the evolution calculation procedure, a series of mean velocity profile measurements were made along the length of the flat plate and the mean flow parameters such as Π, δ, S, ζ and U_τ were obtained by fitting the equation (4).

The experiments were performed in an open blower wind tunnel at The University of Melbourne. The wind tunnel has a settling chamber containing honeycomb and screens. It has a contraction area with a ratio of 8.9:1 that leads into an initial inlet section area of $940mm\times375mm$. The roof of the wind tunnel is fully adjustable to change the pressure gradient accordingly, and the tunnel has a working section length of 6700 mm. For this

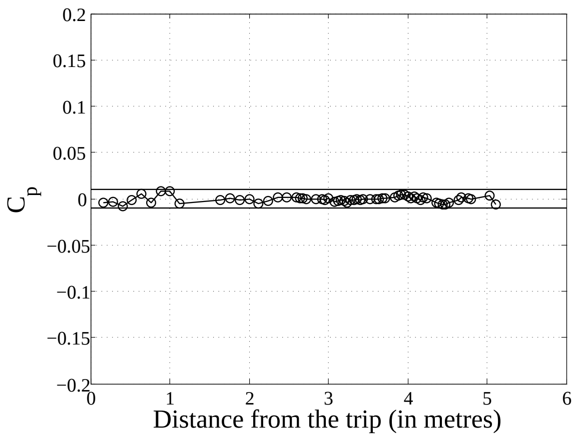

Figure 1: Co-efficient of pressure (C_p) along the wind tunnel .

experiment, the measurements were performed on ZPG, at 16 different streamwise locations. The ZPG pressure gradient was adjusted so that the coefficient of pressure C_p is within ±0.01 throughout the working section as shown in fig (1). Equation (16) shows how to obtain C_p, where p is static pressure measured by the wall tapping, p_0 is the reference static pressure, p_t is the reference total pressure, and U_1 is the free stream velocity.

$$C_p=\frac{p-p_0}{p_t-p_0}=\frac{p-p_0}{\frac{1}{2}\rho U_1^2}=1-\left(\frac{U}{U_1}\right)^2 \tag{16}$$

[2] An equilibrium boundary layer, according to the definitions of Townsend [12, 13] and Rotta [11] requires all mean-relative motions and energy containing components of turbulence (for example, Reynolds shear stress and the turbulent intensities) to have distributions that become invariant with streamwise development when scaeld with local length and velocity scales - see Narasimha & Prabhu [7]

U_∞ (m/s)	x (m)	Π	δ (m)	U_τ (m/s)	ζ	S	l^+	$R_{\delta*}$	R_θ	R_x
9.9076	0.5	0.5934	0.0194	0.43809	0.1509	22.6153	22.5	1815	1237	325252
10.046	0.895	0.7315	0.0257	0.42212	0.2144	23.7979	21.7	2640	1834	585025
10.035	1.425	0.8728	0.0321	0.40014	0.2151	25.0791	20.3	3454	2384	936658
10.004	2.400	0.8887	0.0473	0.38440	0.0066	26.0255	19.5	4944	3447	1585485

Table 1: Mean flow parameters for 4 out of 16 stations are shown. Flow parameters such as Π, δ and U_τ are obtained by fitting (4) on the mean velocity profile. l^+ is the non dimensionalised wire length. $R_{\delta*}$, R_θ and R_x are Reynolds numbers based on displacement thickness, momentum thickness and streamwise distance respectively

Results and Discussion

In this experimental study, a flow case with a free stream velocity of nominally 10 m/s was investigated with a total of 16 mean-flow stations. All of the measurements are performed using single hot-wire constant temperature anemometry (CTA). The sensor used was platinum-wollaston wire of 5μm diameter with an overheat ratio of 1.8. Mean flow parameters for 4 out of 16 stations are shown in Table (1).

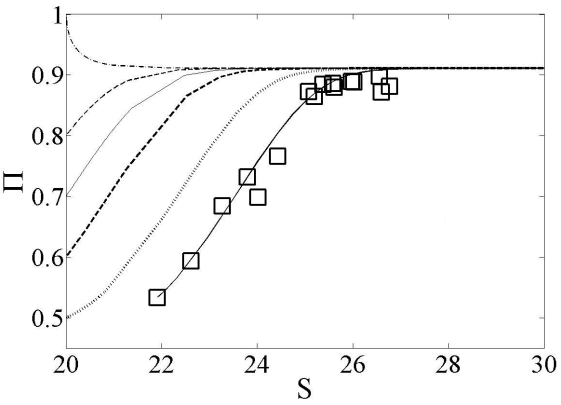

Figure 2: Evolution of wake parameter (Π) for different Non dimensionalised free stream velocity (S). Box (□) symbols represent the experimental data points, the solid line corresponds to evolution calculations with same initial conditions as the experimental data and dotted lines correspond to evolution calculations with different initial conditions.

The shear-stress profile matching technique was used to find the corresponding ζ_a values for each experimental station. In order to calculate the evolution of this flow a functional relationship for (15) needs to be established. Using the experimental data of this study a second order polynomial is proposed for (15)

$$\zeta_a = -4.18\Pi^2 + 5.79\Pi - 1.81 \qquad (17)$$

By using equations (14) and (17), (5) and (6) can now be solved. Figures (2), (3) and (4) show typical calculations for the evolution of ZPG turbulent boundary layers from different initial conditions. From figures (2) and (3), we can see that the ZPG turbulent boundary layer is not an equilibrium layer, at least for finite S, since Π varies with R_x rapidly. Coles [2] notes that for R_θ less than about 2500, Π drops significantly for decreasing R_θ. This effect has been incorporated into the analysis, though this should have a negligible effect on flow at high R_x. Figures (3) and (4) show the evolution of parameters Π and S versus R_x. Figure (5) shows the evolution of Reynolds number based on momentum thickness versus Reynolds based on x. It is obvious that R_θ grows linearly with respect to stream wise Reynolds number (R_x). To help determine the range in which (17) might

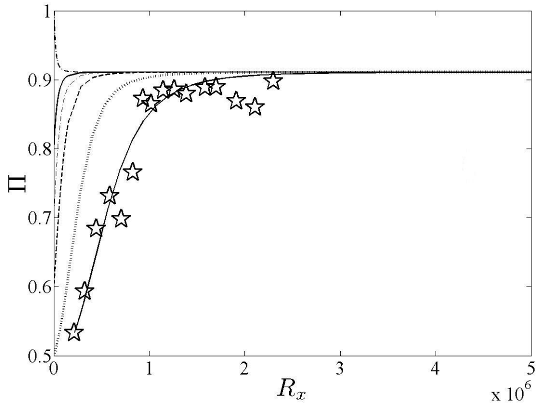

Figure 3: Evolution of wake parameter (Π) for different streamwise Reynolds number (R_x). Star (☆) symbols represent the experimental data points, the solid line corresponds to evolution calculations with same initial conditions as the experimental data and dotted lines correspond to evolution calculations with different initial conditions

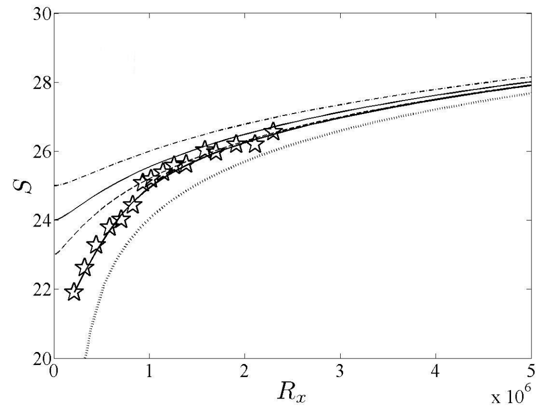

Figure 4: Evolution of skin friction factor (S) for different streamwise Reynolds number (R_x). Star (☆) symbols represent the experimental data points, the solid line corresponds to evolution calculations with same initial conditions as the experimental data and dotted lines correspond to evolution calculations with different initial conditions

be valid, a series of initial conditions were tried. As can be seen in the figures (2) and (3) solution trajectories become constant at about $\Pi \approx 0.9$ as expected. (It is noted that the value of Π depends on the wake formulation used. In this study equation (4) was used. If we use the Coles' [1] wake function, solution trajectories become constant at about $\Pi \approx 0.55$).

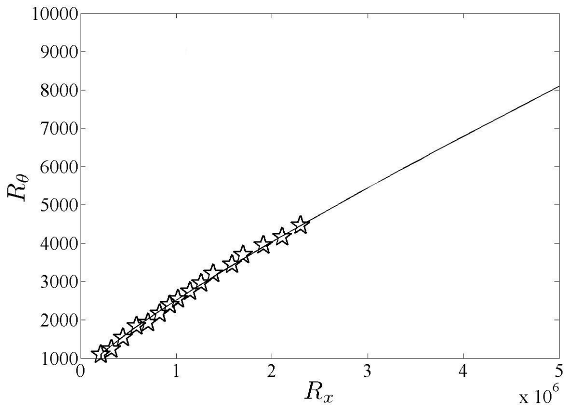

Figure 5: Evolution of momentum thickness Reynolds number (R_θ) for different streamwise Reynolds number (R_x). Star (☆) symbols represent the experimental data points, the solid line corresponds to evolution calculations with same initial conditions as the experimental data.

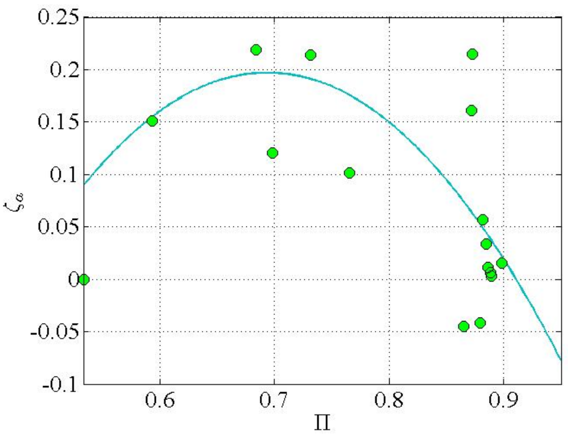

Figure 6: Experimental data of this study. The solid line corresponds to equation (17).

Conclusion

A framework is described that allows us to study streamwise evolution of boundary layers evolving from specific upstream conditions. The framework involves using the law of the wall, the law of the wake, together with the momentum integral and differential equations. The evolution calculation scheme is compared to an experimental ZPG boundary layer evolving over a 6*m* plate and good agreement is observed.

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