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A numerical study of the energy budget in internal bores

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Abstract

Internal bores, or hydraulic jumps, arise in many atmospheric and oceanographic phenomena. The classic single-layer hydraulic jump model accurately predicts a bore's behavior when the density difference between the expanding and contracting layer is large (i.e., water and air), but fails in the Boussinesq limit. A two-layer model, where mass is conserved separately in each layer and momentum is conserved globally, does a much better job but requires for closure an assumption about the loss of energy across a bore. Simple energy relations have been proposed, but none have empirical support due to the difficulties of measuring energy fluxes experimentally. We directly examine the flux of energy within internal bores using 2D direct numerical simulations and find that although there is a global loss of energy across the bore, a transfer of energy from the contracting to the expanding layer causes a net energy gain in that layer. This energy transfer is largely the result of the work performed by viscous stresses at the interface between the two layers. Based on the simulation results, an improved two-layer model is proposed that provides an accurate bore velocity as function of the geometrical parameters, as well as the Reynolds and Schmidt numbers.

Introduction

When a gravity current propagates along an interface instead of a solid boundary, it can be thought of as an internal bore. Internal bores are responsible for many complex and interesting atmospheric and oceanographic phenomena. Perhaps the most visually striking and well known example is the "Morning Glory" cloud formation off the northwestern coast of Australia [4]. Other examples include tidal bores on the Severn river and thunderstorm outflows.

Attempts to model internal bores began with the single layer, inviscid model similar to those used in open channel flow. Later, Yih and Guha[10] proposed a two-layer model be used to improve accuracy. However, their two layer model required making an assumption about the flux of energy through the bore. An internal bore should dissipate energy, but where this dissipation occurs affects the bore's velocity. Early models assumed that all energy dissipation was confined to a single layer [5, 9]. Later, these models were generalized to include dissipation in each layer [7]. These models will be described in more detail below.

In the present investigation, we directly compute the kinetic energy of internal bores through two-dimensional direct numerical simulations in order to gain physical insight into the correct energy jump condition. The results we present here are just a brief overview of our work. A more complete discussion can be found in [2].

Existing theoretical models

The first analytical descriptions of internal bores treated them as single-layer hydraulic jumps. They drew a control volume around the bore (figure 1 top) and conserved mass and momentum across it. The solution of these conservation equations in

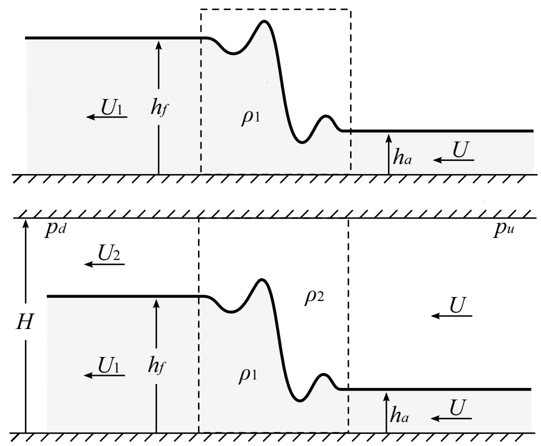

Figure 1: Idealized geometry of a one-layer bore (top) and a two-layer bore (bottom) in a reference frame moving with the bore.

the Boussinesq limit leads to a relation between the bore's front velocity u_f and the geometrical parameters h_a, h_f [6]. Although this relation correctly reduces to the velocity of a gravity wave when $h_f = h_a$, it blows up as $h_a \to 0$, and so is inconsistent in the limit where the bore becomes a gravity current.

As an improvement, Yih and Guha [10] proposed a two-layer model to account for the effects of the upper layer. With the setup in the bottom frame of figure 1, they argued that if the two layers were immiscible, the bore's propagation could be described by conserving mass in each layer and momentum globally. However, these conservation statements do not provide enough information to find the bore's speed. An additional expression is needed to relate the pressure drop along the top of the channel, to the upstream and downstream conditions. Chu and Baddour [3] and Wood and Simpson [9] (hereafter WS) proposed that this fourth expression should be a jump condition relating the up- and downstream energies. Based on dye streak experiments, they argued that energy be conserved in the contracting layer, and applied Bernoulli's equation with no head loss along a streamline in the upper layer allowing a solution for the bore's front velocity. As an alternative, Klemp *et al.* [5] (hereafter KRS) argued that the two-layer model agreed better with available experimental data and gravity current theory if energy was conserved in the lower layer. The authors applied Bernoulli's equation along a streamline in this layer and obtained a different expression for the pressure drop and front velocities. The two expressions for front velocity, written in non-dimensional form using the Boussinesq approximation, are

$$u_{ws} = \left\{ \frac{R\,(1+R)\,(1-Rr)^2}{R^2 r - 3Rr + 2} \right\}^{1/2} \tag{1}$$

$$u_{krs} = \left\{ \frac{R^2\,[2-r\,(1+R)]\,(1-Rr)}{R^2 r - 3Rr + R + 1} \right\}^{1/2}, \tag{2}$$

where $u = U/(g' h_a)^{1/2}$, $R = h_f/h_a$, $r = h_a/H$, and g' is the

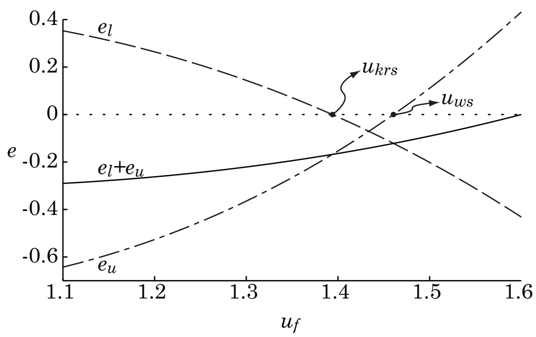

Figure 2: Overall energy loss, and energy change in each layer, given by the LC relation, shown as a function of the front velocity for $R = 2$ and $r = 0.1$. Notice that the WS and KRS front velocities are recovered when e_u or e_l is zero (reprinted from [7]).

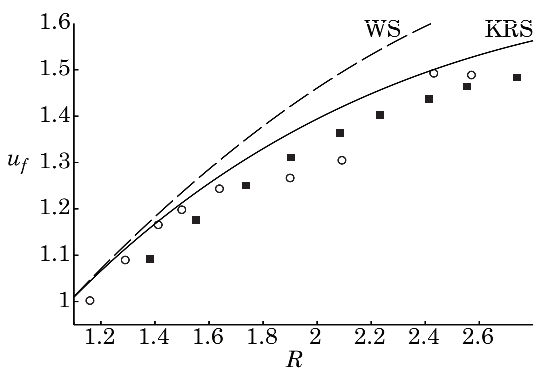

Figure 3: Non-dimensional front velocity plotted as a function of R for $r = 0.1$. Here, the dashed line represents the front velocity predicted by WS, and the solid line corresponds to KRS. The black squares represent our simulation results and the open circles represent experimental data from [9]

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reduced gravity.

Shortly after the publication of the KRS model, Li and Cummins [7] developed a more general relation for the energy dissipated across internal bores that incorporated the WS and KRS models. Instead of assuming that the energy lost across an internal bore is dissipated entirely in one layer, they provide relationships between the energy lost in either the upper (e_u) or the lower (e_l) layer, and the front velocity. Setting e_u or e_l equal to zero recovers the WS or KRS models, respectively, as shown in figure 2. Assuming that none of the two layers can gain energy, the WS and KRS models represent upper and lower bounds on the front velocity.

A comparison of the WS and KRS models with experimental data from [9] and the results of our numerical simulations (figure 3) shows that the KRS model gives better overall agreement, but consistently overpredicts the bore's velocity. Interestingly, figure 2 indicates that front velocities smaller than u_{krs} are associated with a positive e_l, i.e., an energy gain gain of the lower layer across the bore.

Whether or not one of the layers in an internal bore can gain energy is a topic on which there exists some disagreement in the literature. Baines [1] and Wood and Simpson [9] assumed that a gain of energy in one of the layers was unphysical. They argued on this basis against Yih and Guha's original internal bore jump condition (which has not been discussed above) because it implied a slight energy gain in the upper layer. Klemp *et al.* [5] noticed evidence for a small energy gain in the lower layer in some of their modeling studies. They argue that shear at the interface between the two layers could transfer enough energy to the lower layer to overcome dissipation. However, their studies were conducted on a very coarse numerical mesh (300x100) and used a turbulence closure scheme to represent the effects of viscosity, leaving them unable to accurately calculate the energy gain. Finally, Li and Cummins [7] suggest that there is simply not enough experimental data available to support an energy gain in the lower layer and that this issue needs to be studied further.

In the present investigation, we directly examine the flow of energy in internal bores to search for evidence of an energy gain in the lower layer across an internal bore. We expect any potential energy gain to be small, and consequently we employ highly resolved two-dimensional direct numerical simulations for the accurate computation of energy fluxes within an internal bore.

Numerical simulations

To simulate an internal bore, we model a dam-break where a finite reservoir of dense fluid is suddenly released into a two-layer channel. After an initial transient, the dam-break setup produces a steady state bore that can be analyzed by switching to a moving reference frame. We perform 2D direct numerical simulation of the non-dimensional, vorticity-streamfunction formulation of the Navier Stokes equations. With the Boussinesq approximation, they are

$$\nabla^2\psi = -\omega \quad (3)$$

$$\frac{\partial\omega}{\partial t} + \vec{u}\cdot\nabla\omega = \frac{1}{Re}\nabla^2\omega - \frac{\partial\rho}{\partial x} \quad (4)$$

$$\frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho = \frac{1}{ReSc}\nabla^2\rho. \quad (5)$$

We simulate these equations using spectral methods in the streamwise direction and compact finite differences for normal derivatives. An example simulation is shown in figure 4. The code has been validated with three methods: comparing with the experimental data of WS (figure 3), reproducing established results for gravity currents, and checking for overall energy conservation. Details are provided in [2]. Again, for a full description of our governing equations, numerical methods, and algorithms for finding the propagation velocity, local height, and bore height h_f, please see our full manuscript.

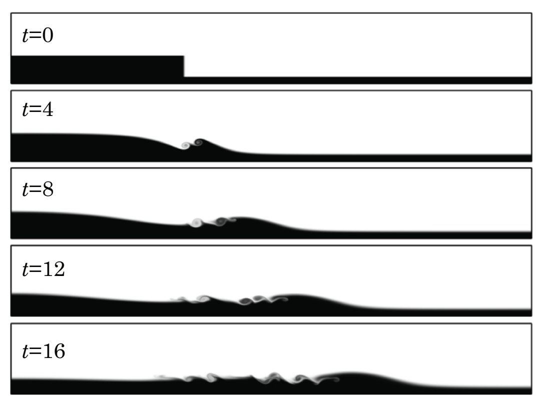

Figure 4: Density field of a representative simulation with snapshots taken every two dimensionless units of time. Here, $R \approx 2.5$, $r = 0.1$, $Re = 3500$, and $Sc = 1$. The left half of the domain is truncated in this figure.

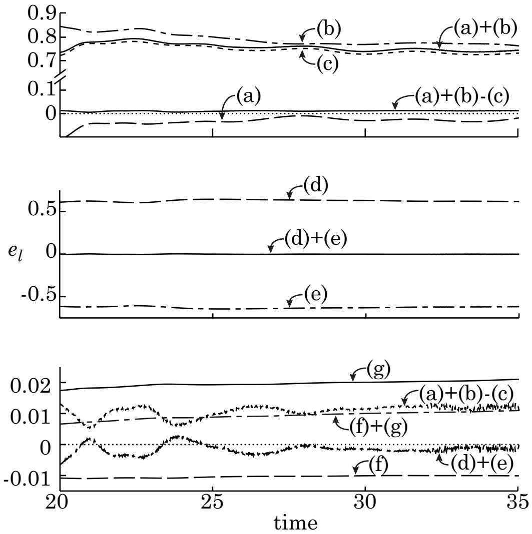

Figure 5: Terms in equation (7) evaluated for a control volume encompassing the lower layer and extending from 20 units behind the bore's front to $x = 55$, well ahead of the bore where all terms in equation (7) are zero. The lines represent the time rate of change of energy (a) and the magnitude of the change of energy due to convection (b), horizontal pressure gradients (c), body forces (d), vertical pressure gradients (e), viscous dissipation (f), and viscous stresses at the interface (g).

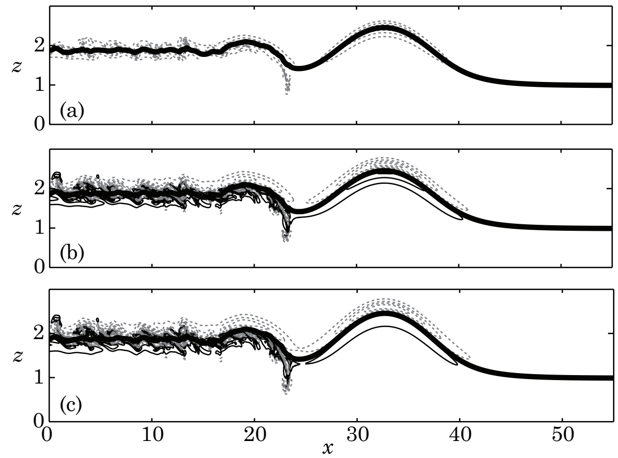

Figure 6: Local time rate of change of E due to (a) viscous dissipation, (b) viscous diffusion of momentum, and (c) both viscous effects. The contours are spaced at even intervals of 0.002 with solid black contours corresponding to positive values and dashed gray ones corresponding to negative ones.

Results

In order to better understand the mechanics of internal bores, we examine a representative case with $R \approx 2$, $r = 0.1$, $Re = 3500$, and $Sc = 1$. All observations of this bore are made at time $t = 32$, well after the bore has reached a quasisteady state.

To investigate the assumptions about the conservation of energy needed for closure in the two-layer models, we track the evolution of the kinetic energy in our simulated internal bores. In the laboratory reference frame, the local time rate of change of kinetic energy is governed by

$$\frac{\partial E}{\partial t} + u \cdot \nabla E = \rho e_g \cdot u + u \cdot \nabla p + \frac{\partial}{\partial x_j}\left(u_i \tau_{ij}\right) + \Phi, \qquad (6)$$

where E is the non-dimensional kinetic energy defined as $u_i^2/2$, τ_{ij} denotes the viscous stress tensor, and Φ represents the viscous dissipation [8]. From left to right, the terms in equation (6) represent the local time rate of change of kinetic energy, the convective flux of kinetic energy, the work performed by body and pressure forces, respectively, the viscous diffusion of momentum (or, equivalently, the work performed by viscous stresses), and the viscous loss of kinetic energy to heat.

Let us consider the evolution of energy in a control volume encompassing our internal bore. To do this, we express equation (6) in integral form as

$$\begin{aligned} \overbrace{\frac{d}{dt}\int E\, dV}^{(a)} + \overbrace{\oint E u \cdot n\, ds}^{(b)} &= \overbrace{\int \rho e_g \cdot u\, dV}^{(d)} + \overbrace{\oint p u \cdot n\, ds}^{(c)+(e)} \\ &+ \underbrace{\oint u_i \tau_{ij} n_j\, ds}_{(g)} + \underbrace{\int \Phi\, dV}_{(f)}. \qquad (7) \end{aligned}$$

Figure 5 shows all of the terms in equation (7) evaluated in a control volume encompassing only the lower-layer. To compare our results with the two-layer models, the flow in the control volume should ideally be in a steady state, so that the time rate of change of kinetic energy in the control volume is zero. Curves (a), (b) and (c) in the top frame of figure 5 suggest that the bore is not in a truly steady state. However, this unsteadiness is primarily due to the interaction of the first three terms in equation (7), as the sum (a)+(b)-(c) displays nearly steady behavior. The unsteadiness does not involve the viscous terms, which are the main focus here.

Figure 5 demonstrates several important points: First of all, it shows that the body force term (d) is cancelled nearly identically by the vertical pressure gradient term (e). This indicates that the hydrostatic assumption is satisfied to a very high degree of accuracy. Secondly, the sum of the time rate of change of energy (a) and the convective flux of energy (b) is almost, but not quite, cancelled by the horizontal pressure gradient term (c). Thirdly, the sum of the inviscid terms (a)+(b)-(c)-(d)-(e)=(f)+(g) is nearly steady, and consistently larger than zero. This indicates that there must be an additional, viscous source of energy that accelerates the fluid. Curve (f) shows that, as expected, dissipation results in a loss of energy. However, the work performed by viscous stresses (g) more than compensates for this loss of energy, and provides a net source of energy for the acceleration of the fluid in the lower layer.

To better understand how the viscous terms are causing the lower layer to gain energy, let us examine the viscous terms in equation (6) locally. Figure 6a shows the local rate of viscous dissipation. Consistent with the analysis by Li and Cummins [7], dissipation occurs in both layers. However, there is also considerable shear between the layers in the bore which causes a diffusion of momentum from the upper to the lower layer as a result of the work performed by the viscous stresses (figure 6b). When these two effects are summed (figure 6c), we recognize that the energy gained by diffusion of momentum across the interface is able to overcome viscous losses in the lower layer, resulting in an energy gain.

In addition to this case study, we conducted a parameter study where we examined the influence of R, r, Re and Sc on the amount of energy gained by the lower layer. We used this parameter study in conjunction with a simple scaling analysis to come up with an improved model for two-layer bores. The full details of our new model are presented in [2], but figure 7 shows

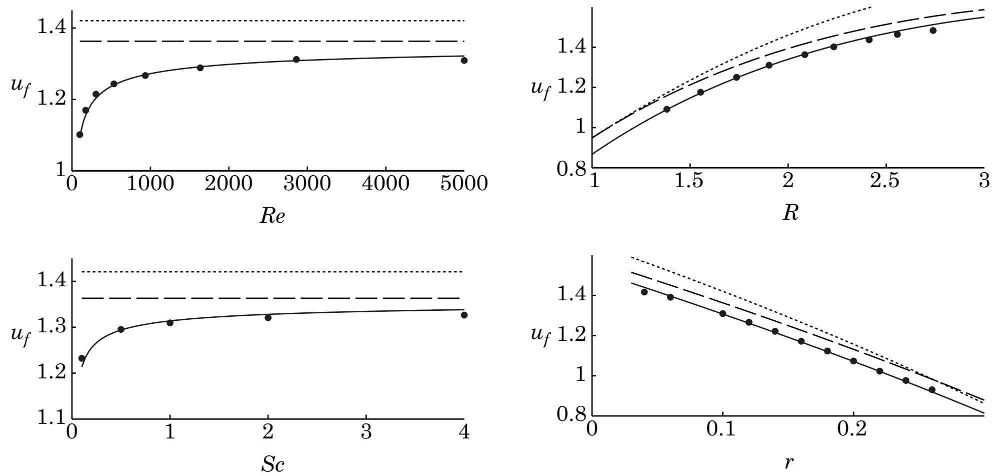

Figure 7: Comparison of the the front velocities in our parameter studies (•) with WS model (dotted line), the KRS model (dashed line), and our new model (solid line).

how our model is more accurate than the WS and KRS models and can account for the effects of *Re* and *Sc* where the other models cannot.

Conclusions

The present investigation employs 2D direct numerical simulations to evaluate two-layer hydrostatic models for internal bores. The main objective was to search for a mechanism whereby the lower, expanding layer of an internal bore might be gaining energy across the bore's front. This would explain why both the WS and KRS models tend to over-predict experimental bore velocities. We conducted a detailed study of a representative bore and found that the viscous diffusion of momentum across the shear layer, i.e., the work of viscous stresses at the interface separating the upper and lower layers downstream of the bore is large enough to overcompensate for the viscous loss of energy in the lower layer, resulting in a net increase of energy in that layer. We then carried out a parameter study and found through a scaling analysis an empirically based expression for the energy gain in the lower layer that, when plugged into the relation developed by Li and Cummins, allows a closed form solution with improved agreement with experimental data.

Acknowledgements

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