

On Discerning Dynamical Structure from the Once-Integrated Momentum Equation

J. Klewicki¹ and P. Fife²

¹Department of Mechanical Engineering
 University of New Hampshire, Durham, New Hampshire, 03824 USA

²Department of Mathematics
 University of Utah, Salt Lake City, Utah, 84112 USA

Abstract

A critical initial step in multiscale/singular perturbation-type studies of the mean flow structure of steady wall-bounded turbulent flows involves determining the appropriately reduced, asymptotically approximate, forms of the mean momentum equation such that they accurately reflect the dynamics over specific subdomains of the flow. Traditionally, such analyses begin with the once-integrated mean momentum equation, rather than its unintegrated form. Simple examples (turbulent Couette, laminar Poiseuille) of flows with known dynamical structure, however, show that physical behavior derived from the once-integrated momentum equation may be ambiguous with regard to ascertaining the dominant dynamical mechanisms. Reasons for this observation and remedies for the shortcomings are discussed. The remedies involve use of the unintegrated mean momentum balance equation, and a new approach to finding the scaling structure of profiles. To clarify specific points, a discussion of turbulent Poiseuille flow is also included.

Introduction

Physical interpretations of the Reynolds averaged momentum equation relating to the mean dynamics of steady turbulent flow near a wall are most often based on its once-integrated form [4, 5, 6, 7]. The objectives of the present effort are to explore the general applicability of this predominant methodology, to point out possible sources of confusion in interpretation, to caution that they may lead to erroneous conclusions, and to suggest alternatives.

In what follows, arguments typically employed in the analysis of wall flows are first brought out in the case of steady turbulent Couette flow. These are shown to lead to possible ambiguities, whereas analysis based on the corresponding unintegrated form of the averaged momentum equation presents a clear picture of the basic physical processes contributing to the flow. The problem of laminar, but high Reynolds number, Poiseuille flow in a channel has a mathematical formulation which is remarkably analogous to turbulent Couette flow. But the usual Couette methodology is shown to lead to fundamental errors in that context. Similarly, turbulent two-dimensional Poiseuille flow has a more complex structure, but the same considerations are shown to apply.

Reasons for the difficulty in properly determining dynamical structure from the once-integrated form of the governing equations are briefly discussed, as are some of the implications of the present findings relative to the classical inner/outer/overlap layer description of turbulent wall-flows. This leads to a brief review of recently developed alternative methodologies and multiscaling approaches.

Specifically, a physically and mathematically justifiable point of view for all these examples is outlined. It consists, at the outset, of determining the relative importance of viscosity, turbulence, and imposed forces in the basic force balance as directly re-

vealed by the unintegrated equation. This sets the stage for applying recently developed criteria for the existence of “scaling patches” to ascertain the scaling structure and overall properties of the mean velocity and Reynolds stress profiles. (Of course all this is trivially known in the case of the laminar example, which is included for illustration and comparison.)

The Standard Methodology

As mentioned above, an intrinsic element of the classical formulations for turbulent wall flow is the use of the once-integrated forms of the momentum equation for the purposes of educing the dominant dynamical mechanisms. To illustrate this, consider fully developed, incompressible, steady (in the mean) turbulent flow between infinite parallel plates with the lower wall at $y = 0$ and the upper wall at $y = 2\delta$. The flow is purely shear driven by the motion of the upper wall in the positive x direction, with velocity $2U_c$ (i.e., the velocity at $y = \delta$ is equal to U_c). Appropriate reduction of the Reynolds averaged Navier Stokes equations gives,

$$0 = -\frac{d\rho\bar{u}v}{dy} + \frac{d\tau}{dy}, \quad (1)$$

where $\tau = \mu dU/dy$ is the mean viscous shear stress. Recall that the dimensions of the terms in this equation are force per unit volume. At this juncture it is relevant to note that equation 1 reveals that the mean dynamics are everywhere determined by an exact balance between the time averaged viscous force and the net mean force due to turbulent inertia. This fact is useful to keep in mind when considering the conclusions derived from the classical analyses that are now presented.

By integrating equation 1, applying the boundary conditions, and defining the integral of first term to be a stress-like quantity $T = -\rho\bar{u}v$, one obtains,

$$\tau(y) + T(y) = \text{const.} = \tau_w, \quad (2)$$

where τ_w is the mean wall shear stress. As is apparent, the dimensions of the terms in this equation are force per unit area or stress. Inner normalization (i.e., by v and $u_\tau = \sqrt{\tau_w/\rho}$, so that $\tau^+(0) = 1$) gives,

$$\tau^+ + T^+ = 1. \quad (3)$$

Neither $\tau(y)$ nor $T(y)$ are analytically derivable. Thus, at this point in the classical analysis relevant flow field data are examined. Figure 1 shows the distributions of τ^+ and T^+ as derived from the DNS data base of Kawamura et al. [3]. As can be seen, $\tau^+ = 1$ at the wall and diminishes to a quantity that is less than $O(1)$ by $y^+ \simeq 30$, i.e., $o(1)$ for all $R_\tau = \delta u_\tau/v$. Conversely, $T^+ = 0$ at the wall, and rises to $O(1)$ by $y^+ \simeq 30$. These behaviors are typically used to provide justification for the classical interpretation of the average structure of the near-wall region

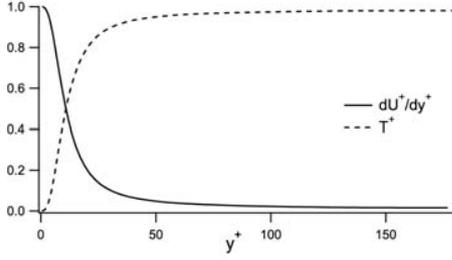


Figure 1: Inner-normalized profiles of τ^+ and T^+ in turbulent Couette flow. Data are from the direct numerical simulation of [3]

given in virtually all treatments of turbulent wall-flows, e.g., [4, 5, 6, 7]. Specifically,

1. in the immediate vicinity of the wall viscous effects are much larger than those associated with turbulent inertia,
2. in an interior region the momentum field mechanisms associated with viscous forces and turbulent inertia are of the same order of magnitude, and
3. for sufficiently large distances from the wall turbulent inertia is dominant.

These regions, of course, correspond respectively to the viscous sublayer, buffer layer, and inertial sublayer plus core region traditionally attributed to turbulent wall-flow structure. With regard to attributing the dominant dynamics to specific physical mechanisms (i.e., viscous versus inertial effects), the conclusions are distinctly at odds with the conclusion drawn earlier relative to equation 1.

Similarly, outer-normalization of equation 2 gives,

$$\frac{1}{R_\tau} \frac{dU^+}{d\eta} + T^+ = 1, \quad (4)$$

where $\eta = y/\delta$. Analyses utilizing equation 4 typically explore its apparent (formal) limiting behavior as $R_\tau \rightarrow \infty$, and arrive at the *outer* approximation, $T^+ = 1$, in the vicinity of $\eta = 1$. This approximation reflects the classical assertion of a turbulent *core region* (see item 3. above) within which the viscous force is negligible. This approximation and equation 3 lead to the inner and outer equations used as starting point for theories based on two scaling regions (outer and inner) plus an *overlap region*. This formulation underlies the Millikan-type derivation of the logarithmic mean profile, as, for example, presented in Schlichting and Gersten [5].

On the General Efficacy of the Standard Methodology

The objective now is to further explore the general efficacy of this methodology by applying it to a flow that has a similar mathematical description, but for which a complete understanding of the dynamics is *a priori* known. Toward this aim, consider fully developed, incompressible, steady laminar flow between infinite parallel plates with the lower wall at $y = 0$ and the upper wall at $y = 2\delta$. The flow is driven in the positive x direction by an applied constant pressure gradient, dp/dx . Appropriate reduction of the Navier Stokes equations gives,

$$0 = -\frac{dp}{dx} + \frac{d\tau}{dy}, \quad (5)$$

where, in this case, $\tau = \mu du/dy$ is the actual viscous shear stress. By integrating equation 5, applying the boundary conditions, and defining the integral of first term to be a stress-like quantity, $P = -\int [dp/dx]dy = -y[dp/dx]$, one obtains,

$$\tau(y) + P(y) = \text{const.} = \tau_w. \quad (6)$$

Inner normalization then yields, as in 3,

$$\tau^+ + P^+ = 1. \quad (7)$$

To attain the stated objective, knowledge of the analytical solution in this laminar flow is momentarily suspended. Instead, the methodology used to educe dynamical structure in the above turbulent flow case is replicated by examining flow field *data*. Profiles of P^+ and τ^+ for this flow are shown in figure 2. As can be seen, $\tau^+ = 1$ at the wall and linearly diminishes to zero at $\eta = 1$. Conversely, $P^+ = 0$ at the wall, and linearly rises to unity at $\eta = 1$. Thus, while the actual functional forms of the stresses (or stress-like quantities) are different for the two cases, the apparent conclusions for the laminar channel flow exhibit clear similarities to those found for turbulent Couette flow.¹ If one were to utilize the identical rationale employed above, the following would again be erroneously concluded:

- In the vicinity of the wall, viscous forces are much larger than those associated with pressure; in an interior region the momentum field mechanisms associated with viscous forces and pressure forces are of the same order of magnitude; and for a region near the channel centerline the applied pressure gradient is dominant.

Similarly, outer-normalization of equation 6 gives,

$$\frac{1}{R_\tau} \frac{du^+}{d\eta} + P^+ = 1. \quad (8)$$

Application of the reasoning employed for turbulent Couette flow leads to the similar conclusion that when $R_\tau \gg 1$ (but not so large that laminarity is unreasonable) there is a *core region* near the center of the channel within which the pressure force is dominant. That is, near the centerline, this reasoning erroneously justifies the simplification, $P^+ = 1$.

Discussion

In the above, two prototypical viscous flows were analyzed using the approach commonly applied to turbulent wall-flows. Later, turbulent Poiseuille flow in a channel will be discussed

¹Note that the validity of a mathematical methodology should be independent of the specific functions involved.

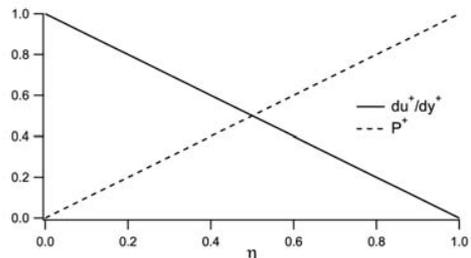


Figure 2: Inner-normalized profiles of τ^+ and P^+ in laminar channel flow.

in this same vein. An essential element of this approach involves the use of the once-integrated form of the momentum equation to identify the dominant dynamical mechanisms in certain subdomains, and from this arrive at appropriately reduced normalized forms of this equation. Not surprisingly, application of this methodology to turbulent Couette flow recovers the set of conclusions typically used to justify the highly accepted inner/outer/overlap scaling region formulation. Application of this same methodology to laminar channel flow, however, leads to a series of conclusions that are verifiably false. The inability of the approach traditionally applied to turbulent wall-flows to correctly recover the known physics of laminar channel flow is interpreted to indicate that this methodology can suggest erroneous conclusions. The reasons for this are now explored further.

An primary element of the above analyses centers on using the basic averaged equations to discern the relative importance of viscosity effects versus turbulence (i.e., inertial) effects. It is evident that this concept is ambiguous without further elaboration. With this aim the following points are offered:

1. Equation 1 is a direct statement that the forces in the channel due to viscosity are everywhere exactly balanced by those due to turbulent momentum transfer. Therefore if, by “viscosity effects” one means forces, then in this flow viscosity effects are always codominant with turbulence effects.
2. The integrated form of these equations, namely 3 and 4, while correct, may direct one to different, even erroneous, conclusions, partly because they express a balance between stresses in the fluid that are integrals of the forces, rather than between the forces themselves. Possible misunderstandings on this point can be resolved by recognizing that the constant “1” on the right of equation 3 represents the stress at the lower wall, which is a consequence of viscosity. Therefore the “viscous” term in that equation should be considered to be $1 - \tau^+$ and the turbulence term T^+ . They balance exactly. Again, with that interpretation of terms and measuring “effect” in terms of stresses, one has that viscosity is everywhere codominant with turbulence. From a point of view near the centerline, the constant on the right of equation 3 represents the effect that a (viscous) stress imposed at a distant point (the wall) has on the stress at the point under consideration. The same is true of the outer-scaled version (the “1” on the right of equation 4). If one excludes that term by considering only local influences due to mean velocity gradients, then it is true that in a region away from the wall, turbulence effects on the stress are dominant over viscous effects.
3. Use of the once-integrated form also includes boundary condition information that does not come into play when considering the unintegrated force balance. This effect is made clear by setting the origin of the channel along the centerline in the laminar flow example, in which case the distribution of P remains linear, but is now zero at the centerline. Clearly the dynamical significance of the viscous or inertial mechanisms should not depend on the choice of coordinate system.
4. The governing balance equations assume many different forms, depending on the choice of scaling applied to the variables. Examples are equations 3 and 4, written using inner versus outer space variables respectively. There are, however, many other possible scalings. In any case, each term in the balance equation can be identified as being derived from either viscous or inertial mechanisms. Any

theoretical treatment of the relative magnitudes of these terms, under any given scaling choice, hinges on the question of whether that scaling is the natural one at the location under consideration. The concept of natural scaling means a scaling of the variable such that derivatives of the dependent scaled variable with respect to the independent ones are $\leq O(1)$. That is, without this condition, it is a certainty that the scaled variables have no chance of maintaining invariant values with increasing R_τ . The natural scaling depends on the location within the flow domain, and changes according to the dominant balance of forces; a particular case being outer scaling.

5. The reduction of equation 4 to $T^+ = 1$ is not obvious unless the first term in equation 4 is actually known to become small when R_τ becomes large, and that fact will follow if η is the natural scaled distance in that region, for then $dU^+/d\eta$ will be bounded independently of R_τ . Although this is the case for Couette and turbulent channel flow, it is not for some analogous flow situations, including the high- R_τ laminar Poiseuille flow described earlier.
6. The correctness of the traditional outer scaling for turbulent Couette or channel flow near the centerline is corroborated by experimental evidence. The only available purely theoretical justification for it lies in the methodology of Fife et al. [1, 2], where this conclusion is obtained along with the correctness of a whole hierarchy of scalings, each with its own scaling domain in the channel. Moreover, in Couette flow each member of the hierarchy reflects a direct balance between the mean viscous force and the mean inertial force owing to the turbulent fluctuations. The union of all the scaling domains covers almost the entire channel; the last and largest member of the hierarchy being the outer scale. Thus, the appropriateness of the outer scale does not arise by neglecting the viscous force, but rather as the natural culmination of a scaling layer hierarchy endowed with the property that within any given member of the layer hierarchy the viscous and inertial forces identically balance.

Some implications of the above are briefly noted. One relates to the fact that equation 1 indicates that there are no subdomains in turbulent Couette flow in which one can rationally neglect the mean viscous shear force versus the mean effects of turbulent inertia, as there is in pressure-driven channel flow. This may be interpreted as there being no classically defined (inertially dominated) outer domain in the Couette case, although “outer” scaling is appropriate in a region near the channel center. Note the two definitions for the term “outer”. Another observation is that the inherent imposition of boundary condition information may underly considerable ambiguity in attempts to discern dynamical structure from the once-integrated momentum equation. Lastly, it is important to emphasize the obvious fact that the correct interpretations of basic dynamical structure related to force balance can always be found from a direct analysis of the unintegrated form of the momentum equation.

In the case of turbulent Couette flow, this analysis is trivial and leads to the ubiquitous balance between forces arising from viscosity and from turbulence. Turbulent Poiseuille flow is more complicated, but in this case the unintegrated mean momentum equation also provides a direct line to the basic physics: data on the relative magnitudes of its three terms, as a function of position in the channel (see figure 1 of [8]), answer the basic question, “Among the three possible types of forces acting on the fluid (viscous, turbulent, and imposed), which ones are the dominant ones, and where?” This knowledge serves to partition

the flow field into four physical “layers” (not scaling layers, but rather regions where specific pairs or triples of forces are in balance).

Further analysis has been performed with a new theoretical framework for turbulent wall flows that is entirely independent of the classical hypothesis of an overlap scaling region [1, 2]. This new theoretical framework, which has been applied to turbulent Couette, Poiseuille, Couette-Poiseuille and several other flow situations, not only reveals the necessary conditions for a logarithmic profile, but also rigorously reveals that these regions are connected by the aforementioned hierarchy of scaling layers, not by an overlapping domain within which the inner and outer scalings are simultaneously valid. In this regard, one element of the classical theory, namely the need for an intermediate coordinate, $\hat{y} = \eta R_+^\alpha$ ($0 < \alpha < 1$), (see [5] p. 523) is retained. Moreover, a theoretical basis is provided for determining the locations and extents of the four physical layers mentioned above.

This new framework employs a reasonable assumed criterion for the existence of “scaling patches”, i.e., regions where certain special scalings (natural scalings) of the variables in the mean momentum equation are valid. It should finally be mentioned that the application of this very criterion to the laminar Poiseuille flow used above to illustrate the inadequacies of classical formal methods, immediately serves to disallow those methods, clarifying why they are inadequate.

Conclusions

For any given problem, the appropriately simplified form of the unintegrated Reynolds averaged Navier-Stokes equation is the time mean differential statement of the Newton’s second law. Thus, for the purposes of educating physics and/or establishing the dominant terms in a multiscale analysis, it is the clearest time-mean expression of the dynamical mechanisms at play. While derived from the unintegrated momentum balance, interpretation of the once-integrated form is not as straight-forward. Though common practice, it is formally incorrect to refer to the once-integrated form as the momentum balance – just as it would be erroneous to similarly refer to the equation for the mean velocity profile (twice-integrated form). The analyses herein indicate that erroneous conclusions may be drawn, relative to flow physics and mathematical structure, if the terms in the once-integrated form are not tracked back to their origin, i.e., to the mechanisms represented in the unintegrated form.

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