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The "zeroth law" of turbulence in steady isotropic turbulence

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Abstract

The dimensionless kinetic energy dissipation rate C_ε is estimated from numerical simulations of statistically stationary isotropic box turbulence that is slightly compressible. The Taylor microscale Reynolds number (Re_λ) range is $20 \lesssim \mathrm{Re}_\lambda \lesssim 300$ and the statistical stationarity is achieved with a random phase forcing method. The strong Re_λ dependence of C_ε abates when $\mathrm{Re}_\lambda \approx 100$ after which C_ε slowly approaches ≈ 0.5, a value slightly different to previously reported simulations but in good agreement with experimental results. If C_ε is estimated at a specific time step from the time series of the quantities involved it is necessary to account for the time lag between energy injection and energy dissipation. Also, the resulting value can differ from the ensemble averaged value by up to $\pm 30\%$. This may explain the spread in results from previously published estimates of C_ε.

Introduction

The notion that the mean turbulent kinetic energy dissipation rate ε is finite and independent of viscosity ν was originally proposed by G. I. Taylor[1]. Its importance is so recognized now that it is sometimes referred to as the "zeroth law" of turbulence. Its existence was assumed by von Kármán and Howarth, Loitsianskii and also, significantly, Kolmogorov[2] in establishing his celebrated similarity hypotheses for the structure of the inertial range of turbulence. Kolmogorov assumed the small scale structure of turbulence to be locally isotropic in space and locally stationary in time - which implies the equality of turbulent kinetic energy injection at the large scales with the rate of turbulent kinetic energy dissipation at the small scales. Although this view should be strictly applied only to steady turbulence, the mechanism of the dissipation of turbulent kinetic energy can be considered the most fundamental aspect of turbulence not only from a theoretical viewpoint but also from a turbulence modeling viewpoint. Indeed, the mechanism that sets the level of turbulent dissipation in flows that are unsteady is a difficult, if not intractable, aspect of turbulence modeling.

The rate of turbulent kinetic energy dissipation is determined by the rate of energy passed from the large-scale eddies to the next smaller scale eddies via a forward cascade until the energy is eventually dissipated by viscosity. Thus, C_ε defined as,

$$C_\varepsilon = \varepsilon L/u'^3, \tag{1}$$

(here, L and u' are characteristic large length and velocity scales respectively) should be independent of the Reynolds number and of order unity. An increase in Reynolds number should only result in an increase in the typical wave number where dissipation takes place. In the past few years there have been a number of numerical (see Ref. [3] and references therein) and experimental (see Refs. [4, 5] for recent results) efforts to determine the value of C_ε and its dependence on the Reynolds number. Perhaps the most convincing of these are the numerical attempts since there is no re-course to one-dimensional surrogacy as there is for experiments. Notwithstanding this fact, there is good agreement, both numerically and experimentally, with the long held view that C_ε is $\sim O(1)$ when the Reynolds number is sufficiently high. The collection of isotropic simulation results for C_ε shown in Ref. [3] indicates that "high enough" Reynolds number "appears" to be $\mathrm{Re}_\lambda \sim O(100)$. Here, $\mathrm{Re}_\lambda (= u'^2[15/\nu\varepsilon]^{1/2})$ is the Taylor microscale Reynolds number. At higher Re_λ e.g. $\mathrm{Re}_\lambda \gtrsim 300$, small Re_λ dependencies for C_ε, such as that proposed by Lohse[6] cannot be ruled out. Measuring such Re_λ dependencies, either numerically or experimentally, will be close to impossible.

One unresolved issue is that raised by Sreenivasan[7]. After assembling all the then known experimental decaying grid turbulence data[8] and numerical data for both decaying and stationary isotropic turbulence he concludes that "the asymptotic value (of C_ε) might depend on the nature of large-scale forcing, or, perhaps, on the structure of the large scale." He also demonstrates[9] in homogeneously sheared flows that the large structure does influence C_ε. However, it might be argued that these results were obtained at low Reynolds numbers and the issue of a universal asymptotic value for C_ε could still be considered open. Alternatively it could be argued that homogeneous shear flows and the like are strictly unsteady turbulent flows and the zeroth law, in its simplest guise, should not be expected to apply to such flows e.g. see Ref. [10]. The possibility of some characteristics of large-scale turbulence being universal should not be ruled out. The recent observation that input power fluctuations, when properly re-scaled, appear universal[11] may be construed to suggest the possibility of universality for C_ε. The aim of the present work is to estimate C_ε from direct numerical simulations (DNS) of statistically stationary isotropic turbulence and compare with previously reported DNS results (summarized in Fig. 3 of Ref. [3]) and experiments carried out in regions of low $(dU/dy \approx dU/dy|_{\max}/2)$ or zero mean shear. The present DNS scheme differs from methods already reported in that a high-order finite difference method is used. To our knowledge, these are the first finite difference results for C_ε. Hence, it is worthwhile to test if different numerics and forcing at the large scales result in vastly different values for C_ε to those already reported.

Numerical Methods

The data used for estimating C_ε are obtained by solving the Navier Stokes equations for an isothermal fluid with a constant kinematic viscosity ν and a constant sound speed c_s. In

Run	N	Re_λ	T_{tot}/T	$\nu\,(\times 10^4)$	$\varepsilon\,(\times 10^5)$	$\Delta t/t_\kappa$	L	λ	u'	$\tau_{\max}/T$	C_ε	η	$k_{\max}\eta$
A	32	20	31	40	24	0.0190	1.9	1.2	0.071	0.15	1.2	0.128	2.1
B	64	42	30	15	22	0.0150	1.6	0.81	0.078	0.37	0.75	0.063	2.0
C	128	90	11	4.0	24	0.0150	1.3	0.43	0.084	0.62	0.54	0.023	1.5
D	256	92	19	4.0	21	0.0071	1.4	0.45	0.081	0.69	0.53	0.024	3.0
E	256	152	20	1.6	21	0.0110	1.4	0.29	0.084	0.74	0.49	0.012	1.5
F	512	219	7	0.80	25	0.0086	1.3	0.20	0.089	0.67	0.47	0.007	1.7

Table 1: Examples of DNS parameters and average turbulence characteristics. N is the number of grid points in each of the Cartesian directions, Re_λ is the Taylor microscale Reynolds number $\equiv u'\lambda/\nu$, T_{tot} is the total run time after the run became statistically stationary, T is the eddy turnover time $\equiv L/u'$, Δt is the run time increment, t_κ is the Kolmogorov time scale $\equiv \nu^{1/2}\varepsilon^{-1/2}$, λ is the Taylor microscale $\equiv u'\sqrt{15\nu/\varepsilon}$, $\tau_{\max}$ is the average time for the energy cascade from large to small scales, and η is the Kolmogorov length scale $\equiv \nu^{3/4}\varepsilon^{-1/4}$.

the numerical simulations the system is forced (stirred) using random transversal waves. The forcing amplitude is chosen such that the root mean square Mach number for all runs is between 0.13 and 0.15 which is not too dissimilar to that found in the wind-tunnel experiments to be discussed in the next section. For these weakly compressible simulations, the energies of solenoidal and potential components of the flow have a ratio $E_{\mathrm{pot}}/E_{\mathrm{sol}} \approx 10^{-4}$–$10^{-2}$ for most scales; only towards the Nyquist frequency (henceforth $k_{\max}$) does the ratio increase to about 0.1. It is thus reasonable to assume that compressibility is irrelevant for the results presented here whilst at the same time the present results can be considered more comparable and relevant to experimental wind tunnel flows than the perfectly incompressible simulations published so far. The code has been validated in previous turbulence studies[12, 13] and the reader is especially referred to Ref.[14] for more information. The simulations are carried out in periodic boxes with resolutions in the range of $32^3 - 512^3$ grid points. The box size is $L_x = L_y = L_z = 2\pi$, which discretizes the wave numbers in units of 1. The viscosity ν is chosen such that the maximum resolved wave number $k_{\max}$ is always greater than $1.5/\eta$, where $\eta = (\nu^3/\varepsilon)^{1/4}$ is the Kolmogorov length scale. To be consistent with previously published DNS studies, the total kinetic energy E is defined as,

$$E_{\mathrm{tot}} = \frac{1}{2}\left\langle \boldsymbol{u}^2 \right\rangle = \frac{3}{2}u'^2 = \int_0^{k_{\max}} E(k)dk, \qquad (2)$$

the integral length scale L is defined,

$$L = \frac{\pi}{2u'^2}\int_0^{k_{\max}} k^{-1}E(k)dk, \qquad (3)$$

and the average turbulent energy dissipation rate is defined as

$$\varepsilon = 2\nu\int_0^{k_{\max}} k^2 E(k)dk. \qquad (4)$$

Angular brackets denote averaging over the box volume. After each run has become statistically stationary (typically 1-2 eddy turnovers $T \equiv L/u'$) the average statistics are estimated for the remaining total run time. Table 1 summarizes the average statistics for each run. Comparing Runs C and D in Table 1 indicates that there is little difference in the average C_ε for simulations resolved up to $\eta k_{\max} = 1.5$ from $\eta k_{\max} = 3$.

Results

Numerical results

In this section results for the higher order finite difference numerical simulations are presented. The simulations began with $N = 32^3$ and each subsequent larger box size began with a velocity field interpolated from the previous box size. Figures

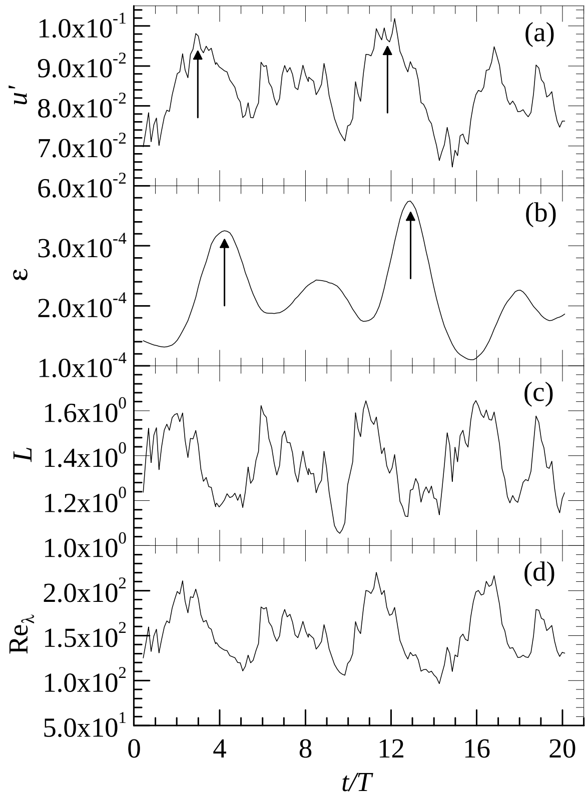

Figure 1: Example time series from Run E, $N = 256^3$, average $\mathrm{Re}_\lambda \approx 152$. (a), u'; (b), ε; (c), L; (d), Re_λ. Here, the eddy turnover time $T = L/u'$. The up arrows $\uparrow$ indicate correlated bursts of u' and ε.

1(a)-(d) show example time series from Run E ($N = 256^3$) for the fluctuating velocity u, the fluctuating integral length scale L, the fluctuating kinetic energy dissipation rate ε and the fluctuating Reynolds number Re_λ respectively. Initially, the turbulence takes a short amount of time to reach a statistically stationary state - a consequence of stabilizing the new run from the previously converged run. The fluctuating quantities shown in Figures 1(a)-(d) are not unlike those encountered in a wind tunnel. Indeed, Fig. 1(a) could easily be mistaken for a hot-wire trace of a turbulent flow. This is in stark contrast to some pseudo-spectral methods that use negative viscosity to maintain a constant energy level e.g. Ref. [3].

Given that the statistics are fluctuating, although they are statistically stationary, it is tempting to plot the instantaneous C_ε as a function of Re_λ. Figure 2 shows C_ε calculated in such a way. The Re_λ dependent trends are obviously not as expected. However, it is worth noting the apparent range for C_ε when $\mathrm{Re}_\lambda \gtrsim 50$ is $\approx 0.3-0.7$ which is the range of previously published DNS results. This may explain the scatter in previously published DNS results if C_ε is calculated from a subjective choice of ε, L and u' at a single time step e.g. as in Ref. [3]. The reason for the incorrect Re_λ dependence for C_ε can be gleaned from Figs. 1(a) and (b). Figure 1(a) shows that an intense burst in turbulent kinetic energy u^2 (an example is noted by the arrow) can be observed some maximum time lag $\tau_{\max}$ later in the turbulent kinetic energy dissipation rate [Figure 1(b), again noted by an arrow]. By noting that there is a strong correlation between intense events of u^2 and L on the one hand and ε on the other hand it is possible to estimate $\tau_{\max}$ from the maximum in the correlation between u'^3/L and ε by

$$\rho_{u'^3/L,\varepsilon}(\tau) = \frac{\overline{[u'^3(t)/L(t)]\;[\varepsilon(t+\tau)]}}{\overline{u'^3(t)/L(t)}\;\overline{\varepsilon(t+\tau)}}\,. \tag{5}$$

With this done for all runs it is possible to shift the time series of $\varepsilon(t)$ for each run by its respective $\tau_{\max}$ and correctly calculate the instantaneous magnitude of C_ε e.g. Fig. 3. Figure 4 shows the newly calculated Re_λ dependence of C_ε using the correct time lag $\tau_{\max}$ for each of the runs. A number of comments can be made. Firstly, the dimensionless dissipation rate C_ε appears to asymptote when $\mathrm{Re}_\lambda \gtrsim 100$. The asymptotic magnitude $C_\varepsilon \approx 0.5$ is in good agreement with the consensus DNS results published so far i.e. $C_\varepsilon \approx 0.4$ to 0.5 (see Ref. [3] and references therein). Having said this and given the present demonstration that it is incorrect to estimate C_ε from a single time snap shot it would be interesting to recalculate previously published results based on subjective choices of the quantities involved for estimating C_ε by using the entire time series. Lastly, the present results verify the use of a high-order finite difference scheme and also prove that the zeroth law applies to slightly compressible turbulence.

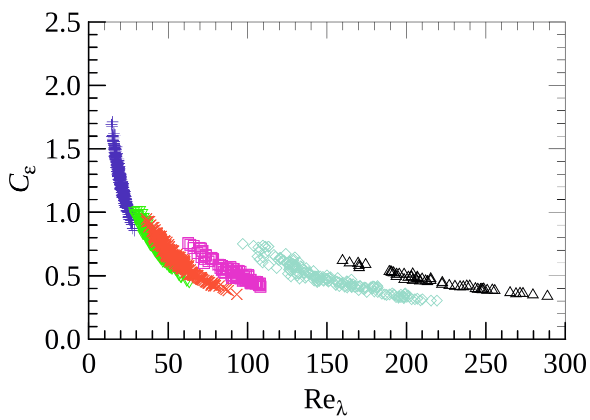

Figure 2: Incorrectly estimated C_ε as a function of Re_λ. +, Run A; ▽, Run B; ×, Run C; □, Run D; ◇, Run E; △, Run F. Ensemble averages can be found in (Table 1).

Experimental results revisited

Results from experiments originally published in Refs. [4, 5], are updated here with more data within the range $170 \lesssim \mathrm{Re}_\lambda \lesssim 1210$. Detailed experimental conditions can be found in Refs. [4, 5] and need not be repeated here. The main group of measurements are from a geometry called a NORMAN grid which generates a decaying wake flow. The geometry is composed of a perforated plate superimposed over a bi-plane grid of square rods. The flow cannot be classed as freely decaying as the extent of the wind tunnel cross section (1.8×2.7 m^2) is approximately $7 \times 11\ L^2$. For all the flows presented in Ref. [4], signals of the fluctuating longitudinal velocity u are acquired, for the most part, on the mean shear profile centerline. For the NORMAN grid, some data is also obtained slightly off the center-line at a transverse distance of one mesh height where $dU/dy \approx dU/dy|_{\max}/2$.

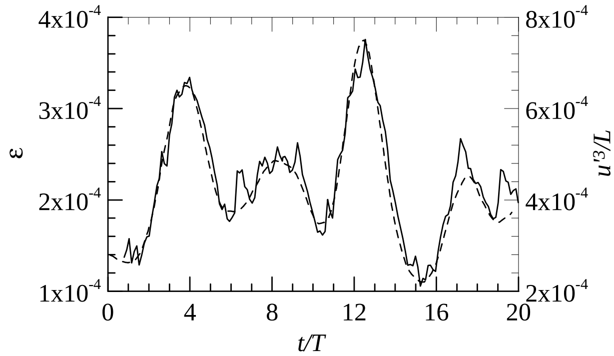

Figure 3: Example of the offset time series for Run E ($\tau^+_{\max} \approx 0.74$), $N = 256^3$, average $\mathrm{Re}_\lambda \approx 150$. Note that the peak events are now well correlated. ——, $u'^3/L(t/T)$; – – –, $\varepsilon([t-\tau_{\max}]/T)$.

All data are acquired using the constant temperature anemometry (CTA) hot-wire technique with a single-wire probe made of 1.27μm diameter Wollaston (Pt-10% Rh) wire. Time lags τ and frequencies f are converted to streamwise distance ($\equiv \tau U$) and one-dimensional longitudinal wave number k_1 ($\equiv 2\pi f/U$) respectively using Taylor's hypothesis. The mean dissipation rate ε is estimated assuming isotropy of the velocity derivatives i.e. $\varepsilon \equiv \varepsilon_{\mathrm{iso}} = 15\nu\langle(\partial u/\partial x)^2\rangle$. We estimate $\langle(\partial u/\partial x)^2\rangle$ from the average value of $E_{1\mathrm{D}}(k_1)$ [the 1-dimensional energy spectrum of u such that $u^2 = \int_0^\infty E_{1\mathrm{D}}(k_1)dk_1$] and from finite differences $\langle(\partial u/\partial x)^2\rangle = \langle u_{i+1} - u_i\rangle^2/(U f_s)^2$. For most of the data, the worst wire resolution is $\approx 2\eta$ where η is the dissipative length scale $\equiv \nu^{3/4}\varepsilon_{\mathrm{iso}}^{-1/4}$. The characteristic length-scale of the large-scale motions L is L_p and is estimated from the wave number $k_{1,p}$ at which a peak in the compensated spectrum $k_1 E_{1\mathrm{D}}(k_1)$ occurs i.e. $L_p = 1/k_{1,p}$.

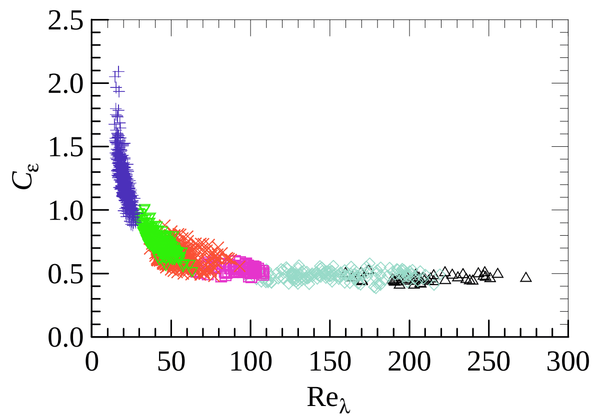

Figure 4: Correctly estimated C_ε as a function of Re_λ. +, Run A; ▽, Run B; ×, Run C; □, Run D; ◇, Run E; △, Run F. Ensemble averages can be found in (Table 1).

Figure 5 shows C_ε for the present data. For all of the data, a value of $C_\varepsilon \approx 0.5$ appears to be the average value. Figure 5 confirms that C_ε, albeit a one-dimensional surrogate, measured in a number of different flows is independent of Re_λ. It could be argued that the rate of approach to an asymptotic value depends on the flow e.g. proximity to initial and boundary conditions. The asymptotic value $C_\varepsilon \approx 0.5$ is in excellent agreement with the present DNS results. These experimental results are encouraging considering that wind-tunnel turbulence is always relatively young compared to DNS turbulence, e.g. the NORMAN grid turbulence has only of the order of 6 eddy turnover times in development by the time it reaches the measurement station.

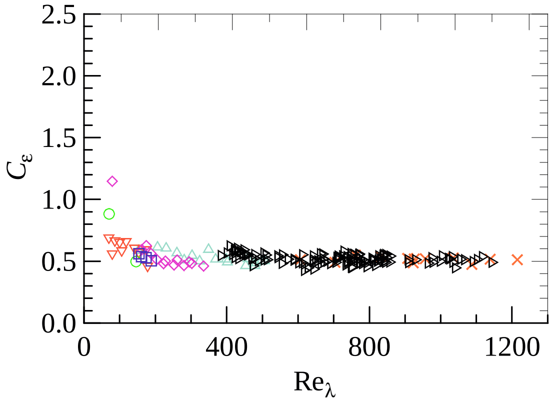

Figure 5: Normalized dissipation rate C_ε for different experimental flows. □, circular disk, $154 \lesssim \mathrm{Re}_\lambda \lesssim 188$; ○, golf ball, $70 \lesssim \mathrm{Re}_\lambda \lesssim 146$; ▽, pipe, $70 \lesssim \mathrm{Re}_\lambda \lesssim 178$; ◇, normal plate, $79 \lesssim \mathrm{Re}_\lambda \lesssim 335$; △, NORMAN grid $N1, 152 \lesssim \mathrm{Re}_\lambda \lesssim 506$; ×, NORMAN grid $N2$ (slight mean shear, $dU/dy \approx dU/dy|_{max}/2$), $607 \lesssim \mathrm{Re}_\lambda \lesssim 1215$,▷, NORMAN grid $N2$ (zero mean shear), $388 \lesssim \mathrm{Re}_\lambda \lesssim 1120$.

Final remarks and conclusions

The present work has revisited the zeroth law of turbulence for both numerical simulations of statistically stationary isotropic turbulence and experiments. The numerical simulations are slightly compressible isotropic turbulence and the statistical stationarity is achieved with a random phase forcing applied at low wave numbers. The main result of the numerical simulations is the demonstration that C_ε should only be estimated with ensemble averaged quantities from the entire time series for which the statistics are stationary. If C_ε is to be estimated at each time snap shot it is necessary to correctly account for the time lag that occurs from the large scale energy injection to the fine scale energy dissipation. Even after correctly correlating the energy injection with the energy dissipation, the instantaneous value of C_ε can vary quite considerably (e.g. $\pm 30\%$) over the extent of the simulation. Such a variation may account for the scatter in magnitude of C_ε in previously published results. Both the present numerical and experimental results suggest that the asymptotic value for C_ε is ≈ 0.5. In light of this, the previously held view that the asymptotic value of C_ε may be dependent on the large scale energy injection could be suspect.

Acknowledgments

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[14] The PENCIL CODE is a cache-efficient high-order finite-difference code (sixth order in space and third order in time) for solving the compressible hydrodynamic and hydromagnetic equations. The code can be obtained from `http://www.nordita.dk/data/brandenb/pencil-code`.