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Analysis of a Transient in a Pipeline with a Leak Using Laplace Transforms

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Abstract

A transient or water hammer event is initiated whenever a steady-state condition in a pipeline is disturbed either by a planned event or accidentally. When the transient reaches a leak, the transient will be reflected and transmitted, which results in a different transient event compared to the transient from the same pipeline without a leak. In a previous study, an analytical solution for the variation in hydraulic grade line (HGL) expressed in terms of a Fourier series has been obtained based on a pipeline with known initial conditions and constant boundary conditions. Based on the analytical solution, a leak detection method was developed previously using leak-induced transient damping. However, The Fourier series approach is not well suited to the case where the boundary conditions vary during the transient event. A Laplace transform solution approach overcomes this difficulty and is the focus of this paper. Normalized hyperbolic governing equations for a pressure transient in a pipeline with a leak are derived, where the discontinuity induced by a leak is considered by using a delta function. In addition the orifice-leak equation is linearized. The accuracy of the analytical solution has been verified by nonlinear numerical analyses using the method of characteristics. The effects of a leak on pipeline transients induced by a pulse boundary perturbation and a continuously changing boundary perturbation are investigated in detail.

Introduction

Pipeline transients, which may be initiated by valve movement, pump shut-down, or change in tank level, are common phenomena in pipeline systems. The pipe flow and pressure transients can be described by a set of non-linear hyperbolic equations derived from the conservation of mass and Newton's second law of motion (conversation of linear-momentum). A closed-form solution for these equations is impossible due to the non-linearity of the momentum equation. A number of methods have been developed to solve these equations analytically where the non-linear term is either neglected [1] or linearized [7, 10], and numerically using the method of characteristics (MOC) and other numerical methods [4, 10]. Predominately, transient pipe flows are studied using one-dimensional models assuming a uniform velocity profile. The neglected two-dimensional or three-dimensional effects are normally approximated by unsteady-friction models [3, 8, 12] with reasonable success.

When a leak exists in a pipeline, the transients are changed compared to the no-leak situation. Recent experimental and numerical work at Adelaide University has demonstrated that attenuation of transients in the pipeline due to leaks is significant. A linear analytical solution for the transients in a pipeline with a leak was derived by applying separation of variables [9]. The analytical solution gives some insight into the mechanism of the effects of a leak on pipeline transients. Based on the analytical solution, a new leak detection method was developed using leak-induced damping on fluid transients [9]. However, that solution was based on constant boundary conditions and known initial conditions, and is not valid for a pipeline with varying boundary conditions. The case of the varying boundary is studied in this paper by applying a transform method, which is more general compared to the traditional eigenfunction methods [5]. The emphasis of this work is to investigate the effects of leakage on pipeline transients. The friction effects can be found in previous studies [2, 7, 10].

A solution based on initial conditions

By linearizing the friction and orifice equations, the governing equation for the transients in a pipeline with a leak can be expressed [9]

$$\frac{\partial^2 h^*}{\partial x^{*2}} = \frac{\partial^2 h^*}{\partial t^{*2}} + [2R + F_L \delta(x^* - x_L^*)] \frac{\partial h^*}{\partial t^*} \tag{1}$$

where $x^* = x/L =$ dimensionless distance, $t^* = t/(L/a) =$ the dimensionless time, L = pipeline length, a = wave speed, $h^* = (H - H_0)/H_1 =$ the dimensionless head of the transient, H = transient head, H_0 = steady-state head, H_1 = a reference head at a tank, $R = fLQ_0/2DAa =$ resistance term, Q_0 = steady-state flow rate, f = Darcy-Weisbach friction factor, D = pipe diameter, A = pipe cross section area, $F_L = C_d A_L a / A\sqrt{2gH_{L0}}$ = leak parameter, C_dA_L = effective leak area, H_{L0} = steady-state head at the leak. $\delta(x^* - x_L^*)$ = Dirac delta function and $x_L^* = x_L/L$= dimensionless leak location.

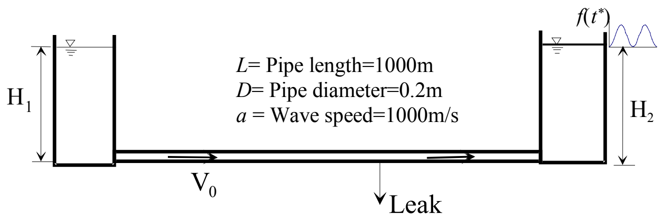

Figure 1 A pipeline connected with two reservoirs

For a pipeline connected between two reservoirs with constant heads as shown in Figure 1, the boundary conditions are given by

$$h^*(0, t^*) = 0 \text{ and } h^*(1, t^*) = 0 \tag{2}$$

and the initial conditions of the pipeline transients may be defined as

$$h^*(x^*,\ 0) = a_{IC}(x^*) \text{ and } \frac{\partial h^*(x^*,0)}{\partial t^*} = b_{IC}(x^*) \tag{3}$$

in which $a_{IC}(x^*)$ and $b_{IC}(x^*)$ are known piecewise continuous functions in the range of $0 \le x^* \le 1$. By applying a Fourier expansion [9], the solution to Eq. (1) subject to boundary and initial conditions expressed by Eqs. (2) and (3) is

$$h^*(x^*, t^*) = \sum_{n=1}^{\infty} \left\{ e^{-(R+R_{nL})t^*} \left[A_n \cos(n\pi t^*) + B_n \sin(n\pi t^*) \right] \sin(n\pi x^*) \right\} \tag{4}$$

where $R_{nL} = F_L \sin^2(n\pi x_L^*) \quad (n = 1, 2, 3, \ldots)$ (5)

is the leak-induced damping factor for harmonic component n. The Fourier coefficients, A_n and B_n, are

$$A_n = 2\int_0^1 a_{IC}(x^*)\sin(n\pi x^*)dx^* \quad (6)$$

$$B_n = \frac{2}{n\pi}\int_0^1 b_{IC}(x^*)\sin(n\pi x^*)dx^* + \frac{(R+R_{nL})A_n}{n\pi} \quad (7)$$

Note that the steady-state friction damping coefficient, R (= $fLQ_0/2aDA$), of Eq. (4) does not depend on n, indicating that the components are exactly exponentially damped by pipe friction, and that the friction damping for all components is equal. In fact, e^{-Rt^*} can be taken outside of the summation in Eq. (4). In contrast, the leak-induced damping factor, R_{nL} of Eq. (5), depends on n and is different for each component; it cannot be removed from the summation sign in Eq. (4). However, leak damping is exactly exponential when applied to a distinct Fourier component and is independent of the measurement position x^* and the transient event. Based on this property a leak detection technique, which is able to detect, locate and quantify a leak in a pipeline, has been developed [9].

A solution for time-dependent boundary conditions

The solution given in Eq. (4) is based on known initial conditions for the pipeline transient regardless of the initiation process of the transient. However, it is more practical to measure the time varying initiation process rather than measure the transient distribution along a pipeline. Due to the limitation of the separation of variables technique in solving partial differential equations with time-dependent boundary conditions, a solution considering the time-dependent initiation process is given using the Laplace transform method.

If a pipeline transient is initiated from a steady state condition by a downstream perturbation process, the governing equation is Eq. (1) and the corresponding boundary and initial conditions are

B.C. $\quad h^*(0,t^*) = 0$ and $h^*(1,t^*) = f(t^*) \quad (8)$

I.C. $\quad h^*(x^*,0) = 0$ and $\dfrac{\partial h^*(x^*,0)}{\partial t^*} = 0 \quad (9)$

in which $f(t^*)$ = dimensionless head at the downstream end of the pipeline.

Pipeline without a leak

For a leak-free pipeline, $F_L = 0$, and applying Laplace transforms to Eq. (1) gives

$$\frac{\partial^2 \tilde{H}(x^*,s)}{\partial x^{*2}} = [s^2\tilde{H}(x^*,s) - sh^*(x^*,0) - \frac{\partial h^*(x^*,0)}{\partial t^*}] + 2R[s\tilde{H}(x^*,s) - h^*(x^*,0)] \quad (10)$$

Considering the conditions in Eq. (9), Eq. (10) is expressed as

$$\frac{\partial^2 \tilde{H}(x^*,s)}{\partial x^{*2}} = s^2\tilde{H}(x^*,s) + 2Rs\tilde{H}(x^*,s) \quad (11)$$

Applying the Laplace transforms to Eq. (8) gives

$$\tilde{H}(0,s) = 0\text{, and } \tilde{H}(1,s) = \tilde{F}(s) \quad (12)$$

in which $\tilde{F}(s) = L\{f(t^*)\}$ = Laplace transform of $f(t^*)$.

For a frictionless pipe, $R = 0$, and the solution for Eq. (11) is

$$\tilde{H}(x^*,s) = C_1 e^{sx^*} + C_2 e^{-sx^*} \quad (13)$$

Substituting Eq. (12) into Eq. (13) and solving gives

$$C_1 = \frac{\tilde{F}(s)}{e^s - e^{-s}}\text{, and } C_2 = \frac{-\tilde{F}(s)}{e^s - e^{-s}} \quad (14)$$

Therefore, Eq. (13) can be expressed as

$$\tilde{H}(x^*,s) = \sum_{n=0}^{\infty}[\tilde{F}(s)e^{-s(2n+1-x^*)} - \tilde{F}(s)e^{-s(2n+1+x^*)}] \quad (15)$$

Applying the inverse Laplace transform to Eq. (15) gives

$$h(x^*,t^*) = \sum_{n=0}^{\infty}\{f[t^* - (2n+1-x^*)]U[t^* - (2n+1-x^*)] - f[t^* - (2n+1+x^*)]U[t^* - (2n+1+x^*)]\} \quad (16)$$

where $U(t)$ is the unit step function defined as

$$U(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases} \quad (17)$$

Pipeline with a leak

For a pipeline with a leak, the pipeline can be considered as two portions divided by the leak with a small neighborhood 2ε as shown in Figure 2.

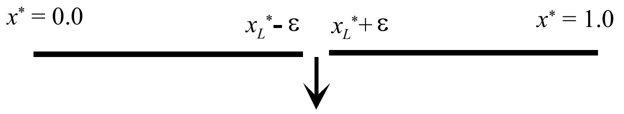

Figure 2 A pipeline with a leak

Integrating Eq. (1) over a small neighborhood on either side of the leak gives

$$\int_{x_L^*-\varepsilon}^{x_L^*+\varepsilon}\frac{\partial^2 h^*}{\partial x^{*2}}dx^* = \int_{x_L^*-\varepsilon}^{x_L^*+\varepsilon}(\frac{\partial^2 h^*}{\partial t^{*2}} + 2R\frac{\partial h^*}{\partial t^*})dx^* + \int_{x_L^*-\varepsilon}^{x_L^*+\varepsilon}F_L\frac{\partial h^*}{\partial t^*}\delta(x^* - x_L^*)dx^* \quad (18)$$

Letting ε approach zero, the first integral on the right hand side of Eq. (18) is zero. Thus Eq. (18) becomes

$$\left.\frac{\partial h^*}{\partial x^*}\right|_{x_L^*-\varepsilon}^{x_L^*+\varepsilon} = F_L\left.\frac{\partial h^*(x^*,t)}{\partial t^*}\right|_{x^*=x_L^*} \quad (19)$$

The governing equations for the two frictionless pipe sections on either side of the leak are

$$\frac{\partial^2 h_1^*}{\partial x^{*2}} = \frac{\partial^2 h_1^*}{\partial t^{*2}} \qquad (0 \le x^* < x_L^*,\ t^* > 0) \quad (20)$$

$$\frac{\partial^2 h_2^*}{\partial x^{*2}} = \frac{\partial^2 h_2^*}{\partial t^{*2}} \qquad (x_L^* < x^* \le 1,\ t^* > 0) \quad (21)$$

For a small leak,

$$h_1^*(x_L^*,t^*) = h_2^*(x_L^*,t^*) \quad (22)$$

Application of Laplace transforms to Eqs. (20) and (21) yields subsidiary equations

$$\tilde{H}_1(x^*,s) = C_1 e^{sx^*} + C_2 e^{-sx^*} \quad (23)$$

$$\tilde{H}_2(x^*,s) = C_3 e^{sx^*} + C_4 e^{-sx^*} \quad (24)$$

Applying Laplace transforms to Eqs. (8), (19) and (22) gives

$$\tilde{H}_1(0,s) = 0 \quad (25)$$

$$\tilde{H}_2(1,s) = \tilde{F}(s) \quad (26)$$

$$\frac{\partial}{\partial x^*}\tilde{H}_2(x_L^*,s) - \frac{\partial}{\partial x^*}\tilde{H}_1(x_L^*,s) = F_L s\tilde{H}_2(x^*,s) \quad (27)$$

$$\tilde{H}_1(x_L^*,s) = \tilde{H}_2(x_L^*,s) \quad (28)$$

Eliminating the four coefficients in Eqs (23) and (24) by solving Eqs. (25), (26), (27) and (28) gives the subsidiary equations

$$\tilde{H}_1(x^*,s)=\frac{-2\tilde{F}(s)e^{s+2sx_L^*+sx^*}}{2e^{2sx_L^*}(1-e^{2s})+F_L(e^{2sx_L^*}-e^{2s})(e^{2sx_L^*}-1)}+\frac{2\tilde{F}(s)e^{s+2sx_L^*-sx^*}}{2e^{2sx_L^*}(1-e^{2s})+F_L(e^{2sx_L^*}-e^{2s})(e^{2sx_L^*}-1)} \tag{29}$$

$$\tilde{H}_2(x^*,s)=\frac{\tilde{F}(s)e^{s+sx^*}(F_L-F_Le^{2sx_L^*}-2e^{2sx_L^*})}{2e^{2sx_L^*}(1-e^{2s})+F_L(e^{2sx_L^*}-e^{2s})(e^{2sx_L^*}-1)}+\frac{\tilde{F}(s)e^{s+2sx_L^*-sx^*}(F_Le^{2sx_L^*}-F_L+2)}{2e^{2sx_L^*}(1-e^{2s})+F_L(e^{2sx_L^*}-e^{2s})(e^{2sx_L^*}-1)} \tag{30}$$

By multiplying both the numerator and denominator by $[2e^{2sx_L^*}(1-e^{2s})]^{-1}$ and rearranging, Eq. (29) is expressed as

$$H_1(x^*,s)=\frac{\dfrac{\tilde{F}(s)(e^{-s+sx^*}-e^{-s-sx^*})}{(1-e^{-2s})}}{1+\dfrac{F_L(e^{2sx_L^*}-e^{2s})(e^{2sx_L^*}-1)}{2e^{2sx_L^*}(1-e^{2s})}} \tag{31}$$

For a small leak, F_L < 1.0, then the second term in the denominator is less than 1.0. By applying a series expansion, Eq. (31) is expressed as

$$H_1(x^*{}'s)=\frac{\tilde{F}(s)(e^{-s(1-x^*)}-e^{-s(1+x^*)})}{(1-e^{-2s})}\sum_{n=0}^{\infty}(-1)^n\frac{F_L{}^n(e^{2sx_L^*}-e^{2s})^n(e^{2sx_L^*}-1)^n}{2^n e^{2nsx_L^*}(1-e^{2s})^n} \tag{32}$$

Applying a series expansion to the denominators of both parts of Eq. (32) gives

$$\tilde{H}_1(x^*,s)=\sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^nF_L^n\tilde{F}(s)}{2^n}\left[e^{-s(1+2i-x^*)}-e^{-s(1+2i+x^*)}\right]\left[\sum_{j=0}^{\infty}(-e^{-s(2+2j-2x_L^*)}+e^{-s(2+2j)}-e^{-s(2j+2x_L^*)}+e^{-s2j})\right]^n \tag{33}$$

Since the product of any two exponential functions is still an exponential function, Eq. (33) may be expressed as

$$\tilde{H}_1(x^*,s)=\sum_{n=0}^{\infty}\frac{(-1)^nF_L^n\tilde{F}(s)}{2^n}\sum_{\substack{i=0\\j=0}}^{\infty}Sign\ [e^{-(a_{ij}-x^*)s}-e^{-(a_{ij}+x^*)s}] \tag{34}$$

in which the values of a_{ij} and *Sign* representing the positive (+) or negative (-) of each term can be determined by Eq. (33). When n = 0, Eq. (34) is the same as Eq. (16) which is the solution for the transients in a pipeline without a leak. Applying the inverse Laplace transform to Eq. (34) gives

$$h_1^*(x^*,t^*)=\sum_{n=0}^{\infty}\sum_{\substack{i=0\\j=0}}^{\infty}\left(\frac{-F_L}{2}\right)^n Sign\{f(t^*-a_{ij}+x^*)U(t-a_{ij}+x^*)-f(t^*-a_{ij}-x^*)U(t^*-a_{ij}-x^*)\} \tag{35}$$

The transients ($0 \le x^* < x_L^*$) in a pipeline with a leak can be calculated using Eq. (35).

Applying a similar procedure, Eq. (30) is expressed as

$$H_2(x^*,s)=\sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^nF_L^n\tilde{F}(s)}{2^n}[(e^{-s(1+2i-x^*)}-e^{-s(1+2i+x^*)})+\frac{F_L}{2}(e^{-s(1+2i-x^*)}+e^{-s(1+2i+x^*)})-\frac{F_L}{2}(e^{-s(1+2i+2x_L^*-x^*)}+e^{-s(1+2i-2x_L^*+x^*)})]\left[\sum_{j=0}^{\infty}(-e^{-s(2+2j-2x_L^*)}+e^{-s(2+2j)}-e^{-s(2j+2x_L^*)}+e^{-s2j})\right]^n \tag{36}$$

An expression of the transient in the portion of the pipeline to the right of the leak ($x_L^* < x^* \le 1.0$) is given as

$$h_2^*(x^*,t^*)=\sum_{n=0}^{\infty}\sum_{\substack{i=0\\j=0}}^{\infty}\left(\frac{-F_L}{2}\right)^n Sign\{[f(t^*-a_{ij}+x^*)U(t-a_{ij}+x^*)-f(t^*-a_{ij}-x^*)U(t^*-a_{ij}-x^*)]+\frac{F_L}{2}[f(t^*-b_{ij}+x^*)U(t^*-b_{ij}+x^*)+f(t^*-b_{ij}+x^*)U(t^*-b_{ij}+x^*)]-\frac{F_L}{2}[f(t^*-c_{ij}+x^*)U(t^*-c_{ij}+x^*)+f(t^*-c_{ij}+x^*)U(t^*-b_{ij}+x^*)]\} \tag{37}$$

where the values of a_{ij}, b_{ij} and c_{ij} are determined by Eq. (36).

Comparison with a non-linear numerical solution

When deriving the governing equation Eq. (1), the nonlinear orifice leak equation was linearized [9]. The accuracy of the linearization is verified in this section by a comparison of the above analytical solution with results calculated numerically based on solving the non-linear equations using the method of characteristics, in which the non-linearity of the friction term is approached by a second-order difference scheme [11].

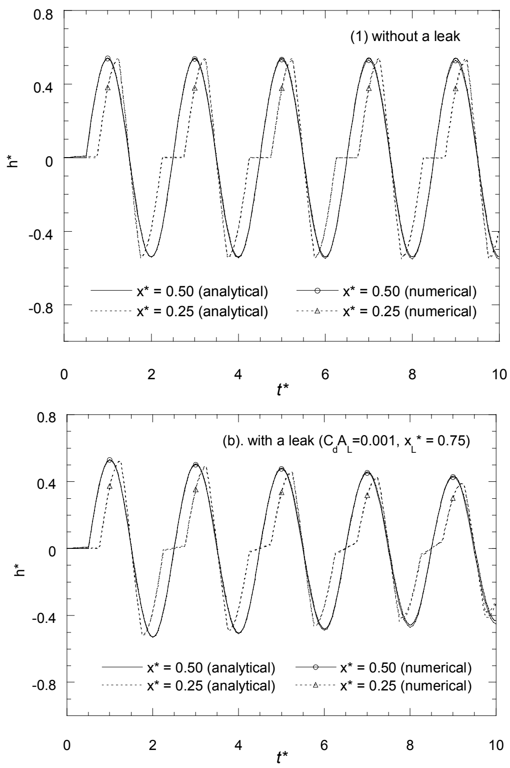

Figure 3 Transients calculated using the analytical solutions and the method of characteristics (numerical and analytical results match almost exactly)

If the boundary condition given in Eq. (8) is a half-sine wave defined as

$$f(t^*) = \begin{cases} 0 & t^* < 0 \\ 0.54\sin(\pi t^*) & 0 \geq t^* \geq 1 \\ 0 & t^* > 1 \end{cases} \tag{38}$$

For the pipeline shown in Figure 1 ($H_1 = H_2 = 25.0$), transients calculated using the numerical method and the analytical solution expressed in Eq. (16) at two measurement locations $x^* = 0.5$ and $x^* = 0.25$ are presented in Figure 3a. When a leak of $C_d A_L / A = 0.001$ is presented at $x_L^* = 0.75$, the transients at the same locations calculated using the numerical method and the analytical solution expressed in Eq. (35) are presented in Figure 3b. The analytical solutions are almost identical to the non-linear numerical results for both cases. In the analytical solution of Eq. (35), only terms of n = 0, 1, 2 were included since the magnitudes of higher terms are very small.

Transients induced by continuously changing boundary conditions

If the boundary condition given in Eq. (8) is a continuously changing boundary perturbation, for example a sine function defined as

$$f(t^*) = \sin(\lambda \pi t^*) \qquad t^* > 0 \tag{39}$$

the transients in a pipeline with and without a leak can be calculated using the analytical solution of Eqs. (16) and (35) (or Eq. (37)). By using two different boundary perturbation frequencies λ = 1.0, and λ = 1.5, the calculated transients measured at $x^* = 0.5$ are plotted in Figure 4.

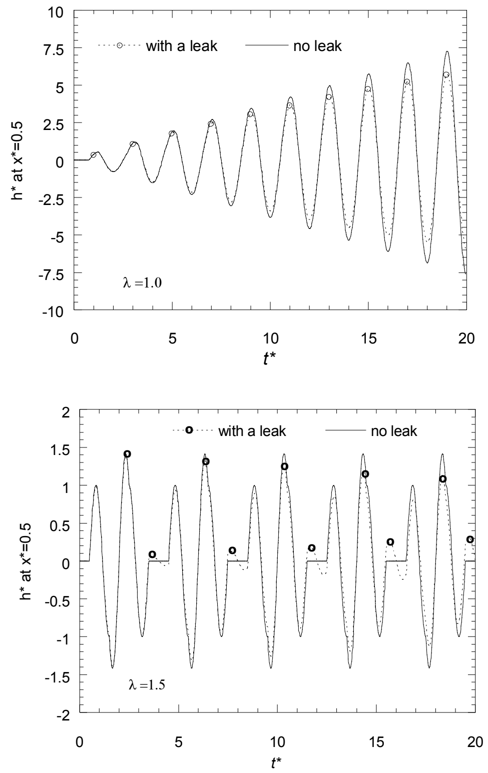

Figure 4 Transients induced by continuously changing boundary conditions using the analytical solution

When λ = 1.0, as the frequency of the boundary perturbation is equal to the natural frequency of the pipeline, a resonance condition is observed. The presence of the leak reduces the amplitude of the resonance. However, it has little influence on the overall shape of the pipeline transient. For the case of with a leak, the transient stops growing and a steady oscillatory flow appears after about 20 periods (t^* > 40.0). When λ = 1.5, a constant periodic transient with a period of T^* = 4.0, is observed in a pipeline without leak. The presence of the leak not only reduces the amplitude of the transient, but also changes the shape of the transient significantly. Compared to the transient in a pipeline without a leak, an additional transient peak is observed. In each cycle, the additional transient peak keeps increasing in magnitude with time while the primary transient peak keeps decreasing. After about 50 periods (t^* > 200.0), the primary and the additional transient peaks reach the same level, and a steady oscillatory flow with a smaller magnitude (about two thirds of the initial primary peak value) appears. Further studies by using different leak locations and leak magnitudes (not included here due to space limit) shows that the magnitude of the steady oscillation under different perturbation frequencies has a unique pattern related to the leak location, and is therefore a good indicator for leak detection.

Conclusions

Analytical solutions of the linearized equations, both with and without a leak, have been derived using a Laplace transform technique that allows for variable boundary conditions. The analytical solution has been shown to be very accurate when compared to the numerical results obtained from the method of characteristics, which include the non-linear friction terms. The analytical solution shows that under a variable boundary perturbation, the influence of a leak on the pipeline transients depends strongly on the frequency of the perturbation. However, the presence of a leak always results in a steady oscillatory flow of a different magnitude compared to the case of without a leak, and the magnitude of the steady oscillatory flow is a good indicator of leak location.

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