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The Structure of Longitudinal Vortices Within the Atmosphere

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Abstract

Curved mixing layers support instabilities in the form of longitudinal vortices aligned in the direction of the flow; these are similar to the Görtler vortices known to exist in curved boundary layers. Vortices of a similar ilk are produced by an unstable temperature stratification. As in Otto, Stott & Denier [8], by making the Boussinesq approximation we may study the heated mixing layer without introducing full compressibility. Our study is aimed mainly at the mixing layer created by a mountain lee wave, and the effects the longitudinal vortices have upon the flow.

If one plots a neutral curve in the wavenumber–Görtler number (G, representative of the level of curvature) plane, two regimes are apparent downstream. One is an inviscid area of growth where $\mathrm{G} \gg 1$ and the spanwise wavenumber is $O(1)$. Here, inviscid vortices develop over short streamwise distances and are governed by a modified form of the Taylor–Goldstein equation. In the second region, known as the right-hand branch, the wavenumber is also large and viscous effects become important.

Introduction

Due to the marked similarities between boundary and mixing layers the underlying theories of the two flows are closely linked; indeed, the only real differences from a mathematical point of view are the boundary conditions.

Görtler [1] showed that the boundary layer present over a concavely curved surface will support counter-rotating longitudinal vortices aligned in the direction of the flow, known to have a spanwise wavelength comparable to the boundary-layer thickness. This important boundary-layer instability was first modelled in a self-consistent manner by Hall [2] who discovered that non-parallel effects in the basic flow are very important and must not be disregarded. When considering the linear theory for vortices with $O(1)$ wavenumbers, the perturbation equations are shown to be parabolic in x (the downstream coordinate) and so can be solved using an Euler marching scheme. Neutral curves can then be generated in the a_x–G_x plane, where a_x is the local wavenumber of the vortices ($ax^{1/2}$) and G_x is the local Görtler number ($\mathrm{G}\chi(x)x^{3/2}$). Here the Görtler number, G, is a dimensionless number that measures the ratio of centrifugal to viscous forces and the function $\chi(x)$ merely contains all the x-dependence of the curvature. These neutral curves are found to be non-unique in the sense that the growth or decay of an imposed disturbance depends both on the method and location of its imposition. However, the right-hand branch of the neutral curve is unique (for a given set of parameters) and an asymptotic solution may be calculated for it.

Hall & Morris [4] made the important discovery that the boundary layer over a heated, yet flat, plate is also unstable to longitudinal vortex structures which develop in the streamwise direction. Whilst akin to Görtler vortices, they are a result of buoyancy rather than centrifugal effects. This discovery led Stott & Denier [13] to study the competing aspects of buoyancy and curvature in the boundary layer. It was found that a large enough convex curvature would overcome the destabilising effect of an unstably stratified temperature profile, and vice versa.

It has been shown experimentally [10] that the curved mixing layer possesses a centrifugal instability similar to the Görtler instability of the boundary layer if the centreline curvature is inclined toward the faster stream. This observation was reproduced theoretically [7], and temperature disparities between the two streams considered [8]. Once more, it was shown that a stably stratified basic flow could support longitudinal vortex instabilities if either sufficiently concave curvatures or free-stream speed disparities were present.

Wherever temperature disparities have been mentioned above, the Boussinesq approximation has been made. This essentially corresponds to ignoring all density variations except in the term of the buoyancy force that includes gravitational acceleration, resulting in the governing equations being roughly the same as in the incompressible case, but with the inclusion of an energy equation and a term $\mathrm{Gr}T$ in the y-momentum equation. Here Gr is known as the Grashof number, a dimensionless quantity which gives the ratio of buoyancy to viscous effects, and $T(x, y, z)$ is the dimensionless temperature. The Boussinesq approximation is widely recognised as being particularly valid in atmospheric applications. Mixing layers have been studied in the fully compressible limit [9, 11] and results are in broad agreement with those obtained using the Boussinesq approximation.

Our intent, therefore, is to study a curved, heated mixing layer in both the inviscid and viscous regimes using the Boussinesq approximation. We choose to concentrate on the mixing layer present within a mountain lee-wave [12]. These occur when the wind blows over a hill or mountain. As the air passes over the obstruction it is forced upwards above its equilibrium position. Descending on the lee side of the hill, the combined effects of buoyancy and gravity cause a damped oscillation. This wave may extend for tens or even hundreds of kilometres, with an amplitude of anything up to about twelve kilometres and a wavelength large enough to be seen on a satellite picture (clouds often form in the peaks of a lee wave, making them visible). The mixing layer occurs between the fast moving air in the wave and the much slower moving air already present on the lee side of the mountain. Obviously curvature plays an important role, arising from the intrinsic curvature of the wave itself, as do differences in temperature between the two air masses and so instabilities of the form described above are expected. Figure 1 is a photograph of a cloud in the peak of a lee wave. The base of this cloud is indented with large features that may be caused by vortices similar to those described above. These structures are very large and could cause a hazard to aircraft. They would be even more dangerous if conditions were not conducive to cloud formation, rendering the vortices invisible to both the naked eye and to conventional radar.

Figure 1: The rippled base of a lenticular cloud situated in a peak of a lee wave created by Table Mountain. The large indentations are possibly created by longitudinal vortex instabilities of the mixing layer, the smaller ripples are caused by a secondary instability.

Formulation

Assume that the temperature of the lower stream at minus infinity is given by T_∞ and the velocity by U_∞; in the case of the mountain lee-wave, these would be $O(10^2\mathrm{K})$ and $O(10\mathrm{m/s})$ respectively. With the addition of a suitable length scale, L, (e.g. the wavelength of the wave, 10^4m) and a mean density ρ_m ($1\mathrm{kg/m}^3$) the flow quantities are non-dimensionalised in the usual manner. Note that a rescaling onto the mixing-layer thickness, $\mathrm{Re}^{-1/2}$, takes place in both the vertical and spanwise directions. This is due to experimental evidence [10], which suggests that the wavelength of the vortices is comparable to the mixing-layer thickness. The Reynolds number, Re, is of the order of 10^9 in this case. Finally, we add a small perturbation to the basic flow and assume that this motion is periodic in the spanwise or z direction.

$$(u, v, w, T) = (\bar{u}, \bar{v}, 0, \bar{T}) + \Delta(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{T})\mathrm{e}^{iaz} + O(\Delta^2),$$
$$p = \bar{p}_0 + \mathrm{Re}^{-\frac{1}{2}}\bar{p}_1(x) + \frac{\mathrm{Gr}}{\mathrm{Re}}\bar{p}(x, y) + \frac{\Delta}{\mathrm{Re}}\tilde{p}\mathrm{e}^{iaz} + O(\Delta^2),$$

where all variables are functions of x and y alone unless otherwise indicated. The quantity Δ is considered to be infinitesimal and the non-dimensional number Gr is known as the Grashof number, given by

$$\mathrm{Gr} = \frac{g\alpha L^3 T_\infty}{\nu^2 \mathrm{Re}^{\frac{3}{2}}}.$$

Here α is the coefficient of volume expansion, g the acceleration due to gravity and ν is the kinematic viscosity of the fluid. For the parameter values given above, the Grashof number takes a value of size $O(10^6)$. Note that we have also rescaled the pressure at this stage, following the scalings suggested by Hall [3]. The use of these pressure scalings allows one to increase the effect of buoyancy on the system; however, we shall not be concerned with this here and will therefore let Gr/Re tend to zero. A third non-dimensional entity is also created by this choice of scaling. This is known as the Görtler number, and is given by $\mathrm{G} = 2\mathrm{Re}^{1/2}\kappa \sim O(10^6)$ where κ is the magnitude of the curvature of the centreline.

The Basic Flow

Taking the leading-order terms only as we let $\mathrm{Re} \to \infty$, we find that the basic flow must satisfy the usual boundary-layer equations, coupled with $\bar{p}_y = \bar{T}$ and a suitable energy equation. This energy equation includes a final non-dimensional number, the Prandtl number (Pr), which is the ratio of momentum diffusivity to heat diffusivity (≈ 0.72 for air at room temperature). However, this will be approximated to unity to allow analytical progress. A similarity solution will now be used to solve this system of partial differential equations, using the similarity variable $\eta = yx^{-1/2}$. The Lock [5] condition will be used as the fifth boundary condition. Two of the three equations may be solved using a Runge–Kutta routine coupled with a secant shooting method, and the pressure may then be solved for using the Trapezium Rule, assuming a reference value at minus infinity. In fact, the form of the pressure profile is fairly arbitrary since it does not affect the perturbation equations at all.

In producing all the results presented here, the free-stream velocity of the upper stream, β_u, was taken as 2, whilst the free-stream temperature, β_t, will take one of two values; 2 for a stably stratified layer or 1/2 for an unstably stratified layer.

Normal-Mode Analysis: Inviscid Modes

In this section we shall consider the stability of the curved, heated mixing layer when the wavenumber of the disturbance, a, is order one and the Görtler and Grashof numbers are both large. In this regime, instabilities develop over a relatively short distance downstream and so viscosity is unimportant. A local normal-mode analysis will be used to determine growth rates of the disturbance in certain parameter regimes.

The important terms (streamwise advection, normal diffusion and buoyancy) must be balanced. Therefore the Görtler number is written as $\mathrm{G} = \mathrm{G}_0\,|\mathrm{Gr}| + \ldots$, and the velocity and temperature components of the disturbance become

$$(\tilde{u}, \tilde{v}, \tilde{T}) = (u_0, |\mathrm{Gr}|^{\frac{1}{2}}\, v_0, T_0)\exp\left(|\mathrm{Gr}|^{\frac{1}{2}}\int \hat{\beta}(x)\mathrm{d}x\right).$$

Here the variables with a subscript zero are functions of the normal coordinate, y, alone. The spatial growth rate is expanded in terms of the Grashof number as

$$\hat{\beta}(x) = \beta_0(x) + |\mathrm{Gr}|^{-\frac{1}{2}}\,\beta_1(x) + \ldots.$$

Using these scalings and taking the limit as $\mathrm{Gr} \to \infty$, the modified Taylor–Goldstein disturbance equation is obtained, namely

$$\beta_0^2\bar{u}^2 v_{0yy} + \left[a^2(\mathrm{G}_0\bar{u}\bar{u}_y + \beta_0^2\bar{u}^2 - \bar{T}_y) + \beta_0^2\bar{u}\bar{u}_{yy}\right] v_0 = 0,$$

where a subscript y denotes a derivative. This equation has been derived previously in the context of the boundary layer [4, 13], and in the mixing-layer case [8]. It is required that the disturbance must die away to nothing at the extremities of the system, therefore the boundary conditions are $v_0 \to 0$ as $y \to \pm\infty$. The system is then solved for the eigenvalue, β_0, using finite-element techniques coupled with a secant shooting method, implementing the normalisation techniques of Otto & Denier [6]. The amount of curvature may be controlled by varying the scaled Görtler number, G_0, and different temperature stratifications may be introduced by changing the value of β_t. This method of varying the upper free-stream temperature is used in favour of changing the sign of the Grashof number (which is chosen to be positive) since it provides control over the degree of stratification as opposed to just whether the flow is stably or unstably stratified. Finally, as mentioned before, the Prandtl number will be taken as unity.

We begin with a stably stratified basic flow, that is one in which warm air lies above cooler air. Figure 2 shows how the growth rate, β_0, changes with the vortex wavenumber, a, for varying degrees of curvature. These growth rates correspond to the most unstable mode in each case. A positive value of the Görtler number corresponds to

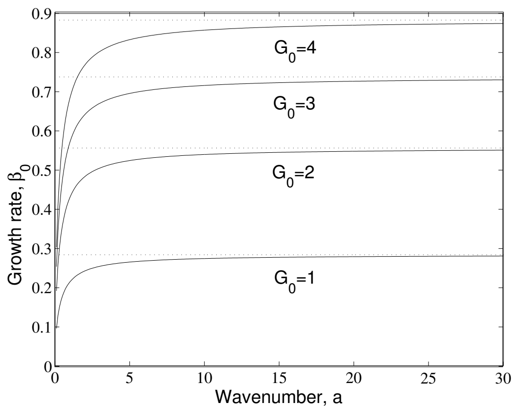

Figure 2: Growth rate vs. wavenumber for a stably stratified basic flow ($\beta_t = 2$) with $\mathrm{Pr} = 1$ and varying degrees of curvature. The dotted lines represent the asymptote to the maximum growth rate in each case [8, 14].

concave curvature, i.e. the centreline curving up into the faster stream, whilst a negative value corresponds to convex curvature. A zero Görtler number, therefore, corresponds to no curvature. No solutions could be found for integer values of G_0 less than one. In fact, it is required that the scaled Görtler number must be greater than one half for longitudinal vortex instabilities to persist for these parameters, [14]. If one compares figure 2 to the results given [8] for the hyperbolic tangent basic-flow profile, one finds that the maximum growth rates are slightly smaller when the similarity solution is used for the basic flow. This disparity increases with the Görtler number, but the results are qualitatively the same in both cases.

Figure 3 again shows how the growth rate, β_0, changes with the vortex wavenumber, a, for varying degrees of curvature. However in this case the basic flow was unstably stratified ($\beta_t = 1/2$). Note that now a solution could be found for $\mathrm{G}_0 = 0$ and that overall the maximum growth rates are considerably larger. Indeed, it may be shown [14] that instabilities are present for all Görtler numbers above $-1/2$. Therefore the unstable stratification destabilises the flow, and is sufficient to overcome the stabilising influence of small convex curvatures.

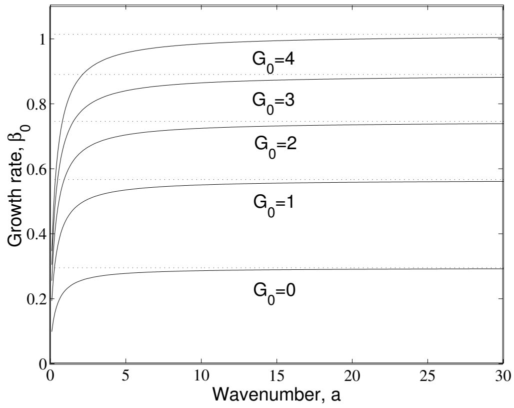

Figure 3: Growth rate vs. wavenumber for an unstably stratified basic flow ($\beta_t = 1/2$) with $\mathrm{Pr} = 1$ and varying degrees of curvature.

Downstream Development of Disturbances

We now consider the downstream development of disturbances at both $O(1)$ wavenumbers (inviscid regime) and large wavenumbers (viscous regime). Beginning with the dimensionless, rescaled governing equations, both the pressure and the spanwise velocity may be eliminated. This leaves a system of equations which are parabolic in the streamwise coordinate, x, opening themselves to a solution via a Crank–Nicolson marching scheme coupled with a standard finite-difference method in the normal (y) direction [7, 14]. An initial disturbance is required to start the scheme off. The form of this is fairly arbitrary [7] since it does not affect the right-hand branch of the neutral curve and so the disturbance chosen is

$$\tilde{u} = [5 + 2(\eta - \bar{\eta})^2]\mathrm{e}^{-(\eta-\bar{\eta})^2}, \qquad \tilde{v} = 0, \qquad \tilde{T} = 0.$$

Here $\bar{\eta}$ is the centre of the imposed disturbance. In the results below the disturbance was placed in the lower stream, with $\bar{\eta} = -5$. The evolution of this disturbance can then be measured via an energy function,

$$E(x) = \int_{-\infty}^{\infty} (\tilde{u}^2 + \tilde{T}^2)\mathrm{d}\eta.$$

From this a spatial growth rate is defined as

$$\sigma(x) = \frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}x} + \frac{1}{2x},$$

and a neutral point is defined as the value of x where $\sigma(x) = 0$, i.e. the transition point between growth and decay. To plot a neutral curve two quantities known as the local Görtler number and local wavenumber are calculated; these take into account the spreading of the mixing layer as we march downstream. During the plotting of the neutral curve these are evaluated at the neutral point. The x-dependence of the curvature, $\chi(x)$, is taken as $\sqrt{x/\bar{x}}$ in all cases. As before, the Prandtl number will be taken as unity, and the upper stream will always be the fastest with $\beta_u = 2$. Once again the stratification of the basic flow will be controlled by varying β_t whilst keeping the Grashof number fixed and positive. Since the limit of a large Grashof (or equivalently Görtler) number has not been taken, the magnitude of Gr becomes important. In the results presented here the Grashof number was chosen to be unity but this may be varied at will.

Here we shall consider only concave curvatures, that is $\mathrm{G} > 0$, although the theory may equally be applied to convex curvatures. Figure 4 shows neutral curves calculated for a range of concave curvatures, with an unstably-stratified basic-flow profile. It is found that, no matter what value of the Görtler number is used, the right-hand branch of the curve always lies in the same position. This fact may also be proved analytically; see [15] for details (in fact the analytical asymptote to the right-hand branch has been included in the figure as a dotted line). It is expected that sustained growth will also be found in

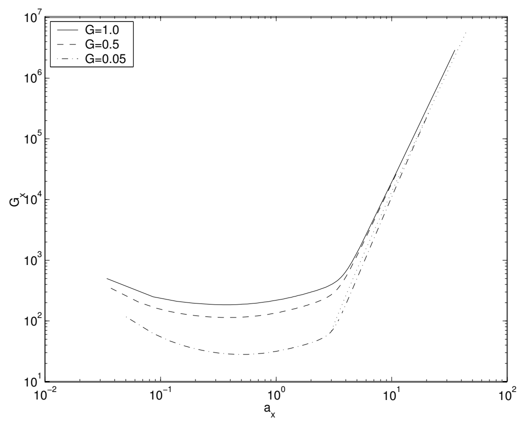

Figure 4: Neutral curves for an unstably-stratified flow and varying concave curvatures.

this system at small levels of convex curvature — these are known as thermal modes [9] since they are driven purely by buoyancy. A similar situation is found when the stratification is stable, although in this case the right-hand branch is moved slightly to the left by the effects of buoyancy (nevertheless the neutral curve still remains 'open'). No thermal modes will be present, however at a small but finite value of the Görtler number growth is no longer sustained as buoyancy overcomes the centrifugal effects, and the right-hand branch moves so far to the left that the curve becomes closed.

Conclusions

We have studied the curved, heated mixing layer in the context of a mountain lee-wave, making use of the Boussinesq approximation. It was seen that when the Görtler and Grashof numbers were large but the non-dimensional wavenumber was order one, the disturbances develop over relatively short streamwise length scales. In this inviscid regime the disturbances are governed by the modified Taylor–Goldstein equation, solved using a similarity solution to represent the basic flow. It was found that a stable temperature stratification can outweigh the destabilising influence of a concave curvature; the converse is also true for an unstable stratification and convex curvature.

The downstream development of disturbances was then studied using a Crank-Nicolson marching scheme in the x-direction. Neutral curves were presented for varying degrees of concave curvatures, and the observation made that the magnitude of the Görtler number has no effect upon the location of the right-hand branch (so long as growth is sustained).

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