14th Australasian Fluid Mechanics Conference
Adelaide University, Adelaide, Australia
10-14 December 2001

Outline of a Theory of non-Rankine-Hugoniot Shock Wave in Weak Mach Reflection

Alfred Siegenthaler(1) and Jehangir Madhani(2)

(1)Swissair, Technical Services TTEC, CH-8058 Zürich-Airport, Switzerland

(2)Dept. of Aeronautical Eng., University of Sydney, N.S.W. 2006, Australia

Abstract

At the previous AFMC, the background for expecting a departure from Rankine-Hugoniot theory at the base of the reflected shock wave in weak Mach reflection was exposed. The results of some pertinent experiments performed in the supersonic wind tunnel were then presented. They confirmed the hypothesised irregular behaviour. In the present contribution, the elaboration of a theory of transgressed shock wave properties is presented. This concept enables to calculate the modified jump process. It fully accounts for the known experimental observations. It is the unyielding boundary conditions that prevail beyond regular reflection which force this remarkable deviation from the classical shock wave theory to take place.

Introduction

The relations describing the changes that take place across shock waves are named after Rankine [17] and Hugoniot [13]. In this paper, classical shock waves shall be referred to as RH-shocks. With regards to the phenomenon of reflecting shock waves, it was the understanding of the mechanism involved in the propagation of strong blast waves over the ground which mainly spurred the research efforts during the WW2 years in the U.S. (see von Neumann [15] and Polachek and Seeger [16]). In Germany, it appears to have been the urge to explain shock branching (also dubbed shock forking) which had been observed in the wind tunnel, and the probable goal was to develop supersonic flight capability (see Weise [20]and Eggink [4]).

While [15, 16] used a plain analytical approach (early access to computing capability?), [20, 4] used the shock polar technique in both forms (hodograph and pressure vs. flow deflection planes) as introduced in 1929 by Busemann [3]. This latter method is best suited to illustrate the problem encountered in matching the boundary conditions of three shock waves meeting at one common point. It enabled [4] and [20] to define a number of boundaries between physically meaningful and complex solutions. In 1964, Henderson [9] succeeded in providing an implicit formulation (polynomial equation of degree 10) to describe the flow in the vicinity of the confluence point of three shock waves. Numerous researchers have contributed over the years in improving the techniques used to unravel the multifaceted aspects of our subject. Very good narrative descriptions of the phenomenon of shock wave reflection have been provided by Bleakney [2], Griffith [7], Hornung [12], Henderson [11] and Ben-Dor [1].

In 1994, Tabak and Rosales [19] endeavoured to solve the inviscid flow equations numerically. Their statements being very clear and to the point, some of their more salient citations are recalled here. Quote:

'... triple shocks, which the equations do not seem to admit, do nonetheless arise. In the context of oblique shock reflections, this is the core of the von Neumann paradox. ... the paradox can not be resolved by invoking viscosity, unless one is willing to admit that the inviscid equations do not have a solution. (...) For small enough *amplitudes* (meaning *shock strengths*), neither regular nor Mach reflection can take place *below this line* (meaning *beyond the detachment point*). The experiments performed ... yielded nonetheless a configuration very much like a Mach reflection, but with shocks that did not seem to satisfy the RH jump conditions. This apparent contradiction is known as the von Neumann paradox of oblique shock reflection. (...) We have also ruled out a whole class of relatively simple 'fan-like' singularities, since these are by nature supersonic, while the region behind a triple point is always subsonic. (...) Unquote.

This last statement refers to Güderley's [8] premise of there being a supersonic patch embedded behind the point of shock confluence. A year and a half ago, Hunter and Brio [14] published a first paper in which they presented a numerical solution based on Burgers' inviscid equations (unsteady transonic small-disturbance equs.) which indeed reveals a minute supersonic flow patch to occur in the wake of the point of shock confluence. This demonstration was reiterated a few months later in a second paper by Zakharian et al. [21] in which the numerical solution was now based on the Euler equations. These are inspiring findings. It must however be realised that these computational studies were performed at a very low incident shock Mach number ($M_i=1.04$). Moreover, the angle of incidence was such that the strength of the reflected wave is very close to the point where it ceases being a shock wave (Mach number of the reflected wave $M_r=1.003$).

The concepts described in the present contribution represent an extension to a preceding paper by the same authors [18]. This shall be referred to hereafter as *part one*. The basic ideas originated from the realisation that in order to set a real chance aside for solving the transition from Regular to Weak Mach Reflection (RR $\rightarrow$ WMR), a fundamentally new approach needed to be ventured.

It was some unexpected observations made many years ago (1978) in the shock tube that led to the experimental work presented in *part one*. A deviation from RH behaviour was presumed to take place in the wake of the reflected shock wave. Calculations suggested that a local drop of stagnation enthalpy might be observed. This called for experiments to be made under steady state conditions. Measurements performed in the supersonic wind tunnel using a mirrored symmetrical wedge model were performed and reported on in *part one*. This set-up generated a reflection pattern as shown in figure 1 (for WMR setting). The testing consisted in recording the stagnation temperature across the wake flow (from here on, the presentation shall be focussed on the ideal gas; the stagnation enthalpy may thus be considered to be proportional to the stagnation temperature, and reference to this latter property shall imply the former). When the flow Mach number was reduced by a small amount in order to be situated slightly beyond RR, the traverse revealed a single trough in the T-stagnation signature. When the Mach number was lowered further, a double symmetrical trough would develop (see figure 4 of *part one*). On schlieren photographs, the shock wave pattern corresponding to the single trough looked deceivingly similar to a RR configuration and was thus labelled Pseudo-RR (PRR). On the other hand, a WMR would show up when the double trough appeared. It was thus demonstrated experimentally that the departure from RH shock behaviour translates into a drop of stagnation enthalpy across the base of the reflected shock wave.

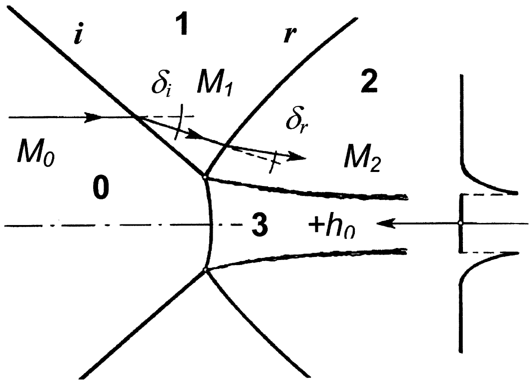

Figure 1: Mirrored weak Mach reflection configuration produced by two symmetrically positioned wedges. This model arrangement has the advantage to generate a 'clean' reflection process that remains widely unaffected by solid boundaries (free of viscous effects). This is an important aspect when investigating the RR ↔ WMR transition regime.

In *part one*, it was further shown that for a given Mach number, the *area* of the trough(s) appears to remain constant with and independent of the distance downstream of the triple point. This observation suggests that the drop of stagnation temperature takes place within the confines of the reflected shock thickness. Moreover, within the distance from the triple point that was looked at, no sign of vortex induced separation of the stagnation temperature profile as reported by Fox et al. [5] was detected. Such a separation, however, could be anticipated to develop further downstream as an effect that would be superimposed on and modulate our drop of temperature.

A further series of experiments was performed in the wind tunnel in August 2000. The aim was to find out whether the departure from RH behaviour observed at the base of a reflecting shock wave could also be witnessed prior to shock detachment at the tip of a wedge. From the physical standpoint, the question is basically the same, namely what occurs after an oblique shock wave has reached is maximum deflection capacity and the ramp imposes yet further streamline deflection. As is well known, the classical understanding is that the shock separates abruptly from the tip as the detachment point is passed. This investigation was again performed using a T-probe that was traversed across the flow field some short distance behind the wedge tip. The scan was started from a hidden position below the ramp surface (inside a trench). The results, while not being as clear cut as was hoped they would be, nonetheless point to the anomalous shock behaviour also taking place at the tip of a wedge. The reason it appears to be somewhat washed out is that, unlike in the symmetrical arrangement used before (ac. figure 1), viscous effects generated on the ramp surface (heat transfer) do interfere with and thus partially mask the drop of stagnation temperature.

In the course of the measurements behind the tip of a wedge, a feature which appears to be similar to the dips (valleys) in the stagnation temperature profiles as reported in a second paper by Fox et al. [6] (see the dips labelled 2 on figures 7 and 8) has also been observed. In [6] these dips were interpreted as cooling produced by unsteady shock-vortex interaction. In our case, it is believed that these were spurious anomalies, for they were produced when the mouth of the probe was crossing the oblique shock front. It ought to be pointed out that the mechanism of the shock-induced total temperature separation reported in [6] is not related in any way to the forced drop of T-stagnation that is being dealt with in the present work.

Hereafter, an outline of a theory which shall enable to calculate the properties of the irregularly behaving reflected shock wave is presented. The transgression the non-RH shock is being subjected to might perhaps prompt the qualification as being a degenerate form of the classical RH shock wave.

Theory

Observations suggest that right behind the shock confluence point, the stagnation temperature profile would look like a furrow as shown on the small diagram drawn on the r.h.s. of figure 1. Moving away from the slip stream out into region 2, the drop of stagnation temperature vanishes and the reflected wave blends into and recovers its RH behaviour. As the downstream distance increases, the furrows (called troughs in *part one*) tend to get dispersed and distorted by the action of the viscous slip stream (shear layer). In this model, the Mach stem remains a classical RH shock (of the strong branch).

As mentioned above, the non-RH transformation forced onto the base of the reflected shock wave occurs within the bounds of the shock thickness. Similarly to classical shock waves, nature strives for thermodynamic equilibrium of the flowing gas to get re-established immediately behind the discontinuity. Our model therefore implies that the drop of stagnation temperature is combined with the thermodynamic jump transformation and both take place inside the confines of the shock thickness.

In *part one* it was pointed out that it is the requisite of RH shock waves being governed by and having to comply simultaneously to the laws of mass, momentum and energy conservation which bars any solution beyond RR. It was demonstrated that in order for the conditions for transition to WMR to be attained, the deadlocked issue is being overcome by forcing the weakest link to yield. And this has been recognised as taking on the form of a transgression of the stagnation enthalpy across the reflected shock. As is well known, the transformation that is induced by a classical shock wave is equivalent to an adiabatic throttling process (Joule-Kelvin effect) where the stagnation enthalpy remains constant. In PRR like in WMR on the other hand, the requirement for a boost of the ailing flow deflection capacity across the reflected shock causes, and in corollary is achieved by a drop of stagnation enthalpy. This observation a priori implies that the requirement of the energy equation is being transgressed (it is noteworthy to observe that Newton's law remains robust in the degenerate process). In mathematical terms, this anomalous behaviour is resolved below by introducing a relaxation coefficient. In order to find solution(s) to the set of equations, the introduction of a new variable is going to call for the formulation of an auxiliary independent relation. The search for an additional equation has proved to be quite a challenging task. The few comments available at this stage shall be taken up later.

Now that the known features (as suggested by experiments) of our degenerate form of shock wave have been delineated, it is proposed to coin a concise designation for it. An appropriate acronym has been found to be ***defect*** shock wave which stands for ***d****eflection* ***e****xtension by* ***f****orced* ***e****nthalpy* ***c****ount* ***t****ranscendence*.

The theory developed below shall enable to calculate the properties of the *defect* shock wave next to the point of shock confluence where the deviation from RH is greatest. The goal is to obtain the analytic jump relations. The shock is treated as a discontinuity and the conservation laws that relate the physical quantities on either side of the shock front, are invoked. The four simultaneous independent equations that govern *defect* shock waves are:

$$\rho_1 u_1 \sin\omega_r = \rho_2 u_2 \sin(\omega_r - \delta_r) \tag{1a}$$

$$p_1 + \rho_1 u_1^2 \sin^2\omega_r = p_2 + \rho_2 u_2^2 \sin^2(\omega_r - \delta_r) \tag{1b}$$

$$\rho_1 u_1^2 \sin\omega_r \cos\omega_r = \rho_2 u_2^2 \sin(\omega_r - \delta_r)\cos(\omega_r - \delta_r) \tag{1c}$$

$$X\left[h_1 + \frac{u_1^2}{2}\right] = h_2 + \frac{u_2^2}{2} \tag{1d}$$

The variable X represents the relaxation coefficient. The choice of the letter was suggested by the fact that the phenomenon is still endowed with many open questions. X depicts the enthalpy conversion ratio across a *defect* shock front. Such shock waves are characterised by $X < 1$. Setting $X = 1$, the RH theory is recovered. The complement $(1-X)$ would represent the enthalpy defect ratio. The angles ω_r and δ_r are the angles of incidence and flow deflection of the reflected shock wave. The gas is considered to be ideal (thermally as well as calorically perfect).

In order to simplify the writing of the forthcoming complex polynomial equations, the following abbreviations are introduced (equs. 2a to 2e):

$$K_1 = (\gamma - 1)M_1^2 + 2 \tag{2a}$$

$$K_2 = \gamma M_1^2 + 1 \tag{2b}$$

$$K_3 = (\gamma + 1)M_1^2 + 2 \tag{2c}$$

$$K_4 = \gamma M_1^2 + 1 - XK_1 \tag{2d}$$

$$K_5 = (\gamma + 1)M_1^2 - XK_1 \tag{2e}$$

The generalised Prandtl velocity relation is worked out first (equs. 3a, b). It is written in a form where the l.h.s. is equivalent to the upstream stagnation enthalpy (sometimes also called reservoir, or total enthalpy). The coefficient B represents the extension for a *defect* shock wave. Setting $B = 1$, the classical Prandtl relation for oblique shocks is recovered.

$$\frac{\gamma+1}{\gamma-1}\cdot\frac{a^{*2}}{2} = B\cdot\left[\frac{u_1u_2\cos\omega\cos(\omega-\delta)}{2} + \frac{\gamma+1}{\gamma-1}\cdot\frac{u_1u_2\sin\omega\sin(\omega-\delta)}{2}\right]$$

$$B = \frac{1 - \dfrac{u_2\sin(\omega-\delta)}{u_1\sin\omega}}{X - \dfrac{u_2\sin(\omega-\delta)}{u_1\sin\omega}} \tag{3a,b}$$

The angle of flow deflection is given by the quadratic (equ. 4):

$$\tan^2\delta_r\,\left(K_3 + XK_1\tan^2\omega_r\right) - 2\tan\delta_r\left(K_4\tan\omega_r - \cot\omega_r\right) - (1-X)K_1 = 0 \tag{4}$$

The angle of incidence is given by the cubic polynomial (equ. 5):

$$\tan^3\omega_r\cdot XK_1 - 2\tan^2\omega_r\cdot K_4\cot\delta_r + \tan\omega_r\left[K_3 - (1-X)K_1\cot^2\delta_r\right] + 2\cot\delta_r = 0 \tag{5}$$

The next two cubic polynomial equations yield the angle ω_e (equ. 6) and the angle ω_s (equ. 7). The first one corresponds to the RR detachment condition (max. flow deflection across the reflected shock) and the second is the condition that yields sonic wake flow ($M_2 = 1$). In both cases, the variables are γ, M_1 and X only:

$$\tan^6\omega_e\cdot XK_1\left[K_4^2 + X(1-X)K_1^2\right] - \tan^4\omega_e\cdot K_4\left[2XK_1 + K_3K_4\right] - \tan^2\omega_e\cdot\left[3XK_1 + 2K_3K_4\right] - K_3 = 0 \tag{6}$$

$$\tan^6\omega_s\cdot XK_1\left[(\gamma+1)M_1^2XK_1 - K_2^2\right] + \tan^4\omega_s\cdot\left\{K_5K_2^2 - 2XK_1\left[(\gamma+1)M_1^2K_4 - \left(M_1^2 - 1\right)\right]\right\} + \tan^2\omega_s\left[(\gamma+1)M_1^2K_4^2 + 2XK_1\left(M_1^2 - 1\right) + K_5\right] + K_5 = 0 \tag{7}$$

With regards to the entropy change across a *defect* shock wave, it may be shown that if expressed as a function of say the static temperature and pressure jumps, the formula remains the same as for RH shock theory. But of course, the T- and p-jumps are no longer tied together by the well known RH shock relation.

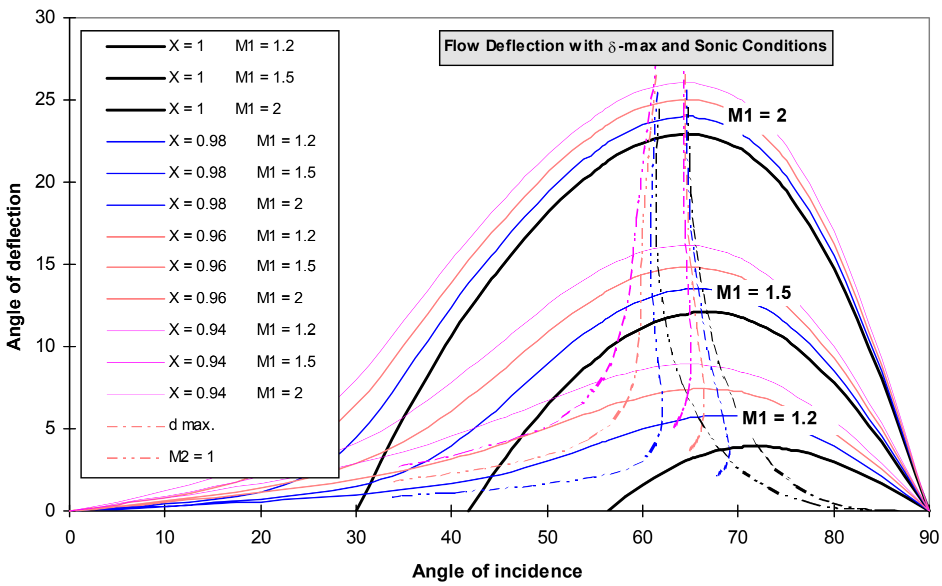

Figure 2:

Flow deflection angle as function of the angle of incidence for Mach numbers of 1.2, 1.5 and 2, each at 4 different values of X. Note that no shock wave is possible for:

$M_1\sin\omega_r < 1$

In the present context, the area of interest is situated along the neighbourhood of the maximum flow deflection capacity.

p_0/p_1	Transition	ω_i	ω_r	δ_r (= δ_i)	M_1	M_2	p_2/p_1	X
0.49	Detachment	42.340	66.170	13.371	1.549	0.920	2.178	1.000
	PRR→WMR	43.132	68.803	13.485	1.517	0.862	2.233	0.983
0.83	Detachment	54.670	72.335	3.658	1.189	0.952	1.331	1.000
	PRR→WMR	58.846	78.872	3.464	1.123	0.832	1.416	0.972
0.905	Detachment	61.398	75.699	1.746	1.111	0.969	1.185	1.000
	PRR→WMR	65.557	81.105	1.572	1.067	0.886	1.238	0.988

Table 1: Calculated points for RR detachment and PRR to WMR transition are shown here. The latter points are the result of an optimisation between the pressure and the enthalpy conversion ratio. The 3 incident shock strengths p_0/p_1 are those used in the experiments as published in Henderson and Siegenthaler [10]. (γ = 1.402)

Discussion

Figure 2 is a display of the solutions of equations (4) to (7). The main feature is to observe that for any Mach number M_1, the flow deflection capacity δ_r increases as X is reduced. A reduction of the enthalpy conversion ratio X across the reflected shock has a twofold beneficial effect with regards to the mechanism for transition from RR to WMR: It produces a rise of the flow deflection capacity coupled with an increase of the pressure ratio. This provides the explanation for the existence of the narrow PRR range: The reflected shock being subjected to a departure from RH behaviour induced by the unyielding boundary conditions, thus manages the imposed flow deflection and sees its pressure rising. Once the compounded pressure ratio across the incident and reflected shock pair matches the pressure jump of a nascent Mach stem, can and does WMR develop.

Pertaining to the missing link needed for solving the set of governing equations (1a) to (1d), the first option that comes to mind is to search for a solution that will minimise the departure from RH behaviour. This idea unfortunately does not provide the correct answers, for the results of such scheme yield falling angles of reflection in the PRR domain whilst experiments clearly show these to be rising. Some form of optimising scheme that involves two, eventually three key parameters appears to grant more success. See calculated results shown in Table 1.

In theory, one could envisage another way the boundary conditions for onset of WMR might be satisfied. In place of the scheme as described (and as observed), the character of the Mach stem could take on values $X > 1$ near the triple point. Obviously, nature makes her choices. One question of interest is whether the scale of the *defect* base of a reflected wave that butts the point of shock confluence does behave self-similarly or not (thinking of experiments in the shock tube or of explosions, for example). No answer to this is available at this stage.

Conclusions

Although the phenomenon of *defect* shock waves presents yet many challenging quests that ought to inspire further research work, a first step in a new direction has been taken. With regards to what certainly represents the major open question, a comment made by Prof. Hans Hornung in the course of a discussion held at the 13th AFMC might best summarise the issue, quote:
Where does the energy go to? It must go somewhere... Unquote.
The answer will most probably not be found with computer modelling, but rather through further investment into experimental work.

S.D.G.

Acknowledgements

Sincere thanks are extended to Prof. Grant P. Steven as well as to Douglas Auld for their continued support for this project. The invaluable help provided by the crew of technicians led by Greg Cumberland and Hasso Nibbe is also gratefully acknowledged.

References

[1] Ben-Dor, G.; Shock wave reflection phenomena, Springer Verlag, 1992.
[2] Bleakney, W.; Review of significant observations on the Mach reflection of shock waves, *Proc. of the 5th Symp. in Applied Maths.*, June 1952, p. 41-47.
[3] Busemann, A.; Verdichtungsstösse in ebenen Gasströmungen, in Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete, 1929. Editors: *A. Gilles, L. Hopf and Th. von Karman; Verlag J. Springer, 1930*.
[4] Eggink, Heinz; Ueber Verdichtungsstösse bei abgelöster Strömung, in *Deutsche Luftfahrtforschung, Forschungsbericht* Nr. 1850, 1943 (Edited by: Aerodyn. Institut der Technischen Hochschule Aachen).
[5] Fox, M. D., Kurosaka, M., Hedges, L. and Hirano, K.; The influence of vortical structures on the thermal fields of jets, *J. Fluid Mech*., **255**, 1993, 447-472.
[6] Fox, M. D., and Kurosaka, M.; Supersonic cooling by shock-vortex interaction, *J. Fluid Mech*., **308**, 1996, 363-379:
[7] Griffith, W. C.; Shock waves, *J. Fluid Mech.*, **106**, 1981, 25th Anniversary Issue, 81-101.
[8] Güderley, K. G.; *The Theory of Transonic Flow,* Pergamon Press, Oxford, 1962.
[9] Henderson, L. F.; On the confluence of three shock waves in a perfect gas, *Aero. Quart.* **15**, May 1964, 181-197.
[10] Henderson, L.F. and Siegenthaler, A., Experiments on the diffraction of weak blast waves: the von Neumann paradox, *Proc. R. Soc. Lond*., **A369**, 1980, 537-555.
[11] Henderson, L. F.; Regions and boundaries for diffracting shock wave systems, *ZAMM*, **67**, 1987, 73-86.
[12] Hornung, H. G.; Regular and Mach reflection of shock waves, *Ann. Rev. of Fluid Mech.*, **18**, 1986, 35-58.
[13] Hugoniot, P.-H.; Mémoire sur la propagation du mouvement dans les corps et plus spécialement dans les gas parfaits, *J. de l'École Poly-technique*, **57**: 3-97 (1st part, 1887), **58**: 1-125 (2nd part, 1889).
[14] Hunter, J. K., and Brio, M.; Weak shock reflection, *J. Fluid Mech.* **410**, 2000, 235-261.
[15] von Neumann, J.; Oblique Reflection of Shocks, etc., 1943, see *John von Neumann- Collected Works IV*, Pergamon Press, 1963.
[16] Polachek, H., & Seeger, R. J.; On shock wave phenomena: Interaction of shock waves in gases, *Proc. of Symposia in Applied Maths.*, **1**, 1949, 119-144.
[17] Rankine, W. J. M.; On the thermodynamic theory of waves of finite longitudinal disturbance, *Phil. Trans. (of the R. Soc. of London)*, **160/II**, 1870, 277-288.
[18] Siegenthaler, A. and Madhani; J., Experimental verification of departure from Rankine-Hugoniot behaviour in weak Mach reflection, *13th Australasian Fluid Mechanics Conference*, Monash University, Melbourne, 1998, 549-554.
[19] Tabak, E. G., and Rosales, R. R.; Focusing of weak shock waves and the von Neumann paradox of oblique shock reflection, *Phys. of Fluids*, **6**(5), 1994, 1874-1892.
[20] Weise, A.; Theorie des gegabelten Verdichtungsstosses, in *Jahrbuch der deutschen Luftfahrtforschung, Zentrale für wissenschaftliches Berichtwesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB)*, 1943, Berlin-Adlershof, Anl. IIA 011.
[21] Zakharian, A. R., Brio, M., Hunter, J. K. and Webb, G. M.; The von Neumann paradox in weak shock reflection, *J. Fluid Mech.* **422**, 2000, 193-205.