14th Australasian Fluid Mechanics Conference
Adelaide University, Adelaide, Australia
10-14 December 2001

Internal-wave radiation from an oscillatory source in a depth-dependent medium.

Dave Broutman[1] **and James W. Rottman**[2]

[1]Computational Physics, Inc.
Springfield, VA, 22151, USA

[2]Science Applications International Corporation
San Diego, CA, 92121, USA

Abstract

We develop a method for analysing linear internal waves generated by an oscillating obstacle moving horizontally at constant speed through a depth-dependent medium. The new feature of the method is that it uses ray theory not to approximate the wavefield directly (which would fail near the source and near caustics) but to approximate the normal modes. The normal modes are then combined by inverse Fourier transform to obtain a spatial description of the wavefield. We discuss this method and its relation to the theory of Lighthill [5]. We illustrate the method by computing lee waves radiated from a sphere in a cross flow with vertical shear.

Introduction

We examine the propagation of internal waves generated by an obstacle moving horizontally at constant speed while oscillating vertically at constant frequency. The mean flow and mean buoyancy are depth dependent, and the linear solution is computed using a Fourier integral representation. This is a standard approach that requires the calculation of (vertical) normal modes. For a non-uniform medium, the normal modes are generally difficult to obtain in a useful analytical form, so we make a simplifying modification. Instead of computing the normal modes exactly, or numerically, we approximate them with ray theory. The ray approximation for the normal modes is easy to obtain, and when substituted into the integral representation the result can give a reasonable approximation of the linear wavefield. This includes regions near caustics and in the vicinity of the obstacle, where traditional ray methods fail.

We refer to this approach as Maslov's method, after the Russian V.P. Maslov who devised something similar to correct singularities in the semi-classical limit of quantum mechanics [6]. In the next section, we derive Maslov's solution for our problem. The solution requires a reference-level value at some depth near the obstacle. We use a theory of Lighthill [5] to approximate the solution at this depth. We then apply Maslov's method to lee waves generated by a sphere in a cross current with constant vertical shear.

Maslov's Method

We consider a stratified Boussinesq fluid. The coordinate system, $\mathbf{r} = (x, y, z)$ with z positive upwards, is fixed to the mean position of the oscillating obstacle. The internal waves have wavenumber vector $\mathbf{k} = (k, l, m)$. The buoyancy frequency is $N(z)$, and the mean flow is $(U(z), V(z), 0)$.

The internal-wave dispersion relation is

$$m^2 = (k^2 + l^2)(N^2 - \hat{\omega}^2)/\hat{\omega}^2, \tag{1}$$

where $\hat{\omega} = \sigma - kU - lV$ is the intrinsic frequency. We seek solutions of constant σ, the frequency of the source.

Let w be the vertical velocity and consider a normal-mode solution of the form

$$\hat{w}(k,l,z)e^{i(kx+ly-\sigma t)}. \tag{2}$$

A model equation for $\hat{w}$ consistent with the dispersion relation and suitable for our purposes is

$$\hat{w}_{zz} + m^2(z)\hat{w} = 0. \tag{3}$$

For $V = 0$, (3) reduces to the Taylor-Goldstein equation, except for a U_{zz} term in the latter that can be ignored in the ray approximation. The ray solution to (3) is (Gill [4], Section 8.12)

$$\hat{w}(k,l,z) = \hat{w}_0(m_0/m)^{1/2} e^{i\int_{z_0}^{z} m(k,l,z')dz'}. \tag{4}$$

Here $\hat{w}_0 = \hat{w}(k,l,z_0)$ and $m_0 = m(k,l,z_0)$ are values at some reference level z_0.

The Maslov solution w_m is the Fourier integral of (4)

$$w_m(\mathbf{r},t) = e^{-i\sigma t} \iint_{-\infty}^{\infty} \hat{w}_0(m_0/m)^{1/2} e^{i\int_{z_0}^{z} m(k,l,z')dz'} e^{i(kx+ly)} dkdl \,. \tag{5}$$

Note that the exact normal modes are not required. The integrand depends on the ray solution for m and values at the reference level value z_0 (see next section). The integral can be approximated by Fourier series and calculated efficiently with an FFT.

To express Maslov's solution in another variable, we simply use ray theory to relate that variable to $\hat{w}$. For example, $D\eta = w$, where η is the vertical displacement and $D = \partial_t + U\partial_x + V\partial_y$ is the linearised advective derivative. With $\hat{w} = -i\hat{\omega}\hat{\eta}$, we have

$$\eta_m(\mathbf{r},t) = e^{-i\sigma t} \iint_{-\infty}^{\infty} \hat{\eta}(k,l,z)\; e^{i(kx+ly)} dkdl \tag{6}$$

$$= e^{-i\sigma t} \iint_{-\infty}^{\infty} \hat{\eta}_0(m_0/m)^{1/2}(\hat{\omega}_0/\hat{\omega}) e^{i\int_{z_0}^{z} m(k,l,z')dz'} e^{i(kx+ly)} dkdl \,. \tag{7}$$

As before, the subscript zero means evaluation at $z = z_0$.

Eq. (7) is equivalent to the result derived by Broutman *et. al.* [1] from wave-action conservation, in a way similar to Brown [2]. Broutman *et. al.* analysed stationary mountain waves in wind shear and showed that Maslov's method agrees well with a traditional integral method involving normal modes. In the mountain-wave problem, the depth z_0 corresponds to the ground, where a (linearised) boundary condition is applied. In the present problem, the depth z_0 does not correspond to a boundary. We approximate the solution at z_0 using a theory from Lighthill [5], Section 4.9

Lighthill's theory describes wave radiation from an oscillating moving source, assuming a uniform medium. If we choose z_0 near the source, then we can justify the uniform-medium assumption because refraction is relatively unimportant in this region. The predominant process affecting wave propagation is the strong divergence of waves away from the source, and this process is accounted for by Lighthill's theory.

In the following, for brevity, we will derive and plot solutions only for the vertical displacement η.

Lighthill's solution at z_0

Lighthill [5] considers forcing of a general linear system by an oscillatory source of frequency σ and spatial distribution $f(\mathbf{r})$. His model equation is of the form

$$L\eta(\mathbf{r},t) = f(\mathbf{r})e^{-i\sigma t} \tag{8}$$

where L is a linear constant-coefficient partial-differential operator. By regarding $\partial_t = -i\sigma$ and $\nabla = i\mathbf{k}$, L can be written as a function B of σ and $\mathbf{k}$, with

$$B(\sigma,\mathbf{k}) = 0 \tag{9}$$

defining a dispersion relation and a wavenumber surface for frequency σ. Lighthill applies a Fourier transform to (8) and then approximates the result with a stationary-phase analysis. He finds that

$$\eta(\mathbf{r},t) = \frac{4\pi^2 F}{|\nabla_{\mathbf{k}} B|\ |\kappa|^{1/2} r} e^{i(kx+ly+mz-\sigma t+\gamma)}. \tag{10}$$

Here r is the distance from the origin, and κ is the Gaussian curvature of the wavenumber surface defined by (9). The gradient $\nabla_{\mathbf{k}} B$ is with respect to the wavenumber vector $\mathbf{k}$. The phase-shift γ depends on the principal curvatures of the wavenumber surface (see [5]). F is the Fourier transform of f, defined such that

$$f(\mathbf{r}) = \iiint_{-\infty}^{\infty} F(k,l,m)e^{i(kx+ly+mz)}dkdldm. \tag{11}$$

For group velocity $\mathbf{c_g} = (c_{g1}, c_{g2}, c_{g3})$ the stationary-phase condition

$$x = zc_{g1}/c_{g3} \qquad y = zc_{g2}/c_{g3}, \tag{12}$$

relates the wavenumber in (10) to position, yielding a spatial description of the wavefield.

We want to use Maslov's solution instead of (10) because (10) breaks down near caustics. Maslov's integrand (7) requires a reference-level solution expressed in a normal-mode representation with coordinates k,l,z_0. We obtain this by reverting to Lighthill's full integral representation without stationary-phase approximation. As follows from (8) and subsequent equations, this is (cf. (269) of [5])

$$\eta(\mathbf{r},t) = e^{-i\sigma t} \iint_{-\infty}^{\infty} I\, e^{i(kx+ly)}dkdl. \tag{13}$$

where

$$I = \int_{-\infty}^{\infty} (F/B)\, e^{imz}dm. \tag{14}$$

Maslov's solution (for a uniform medium) is obtained by evaluating I asymptotically and then substituting the result into (13). The asymptotic form for I gives its ray approximation and is the uniform-medium limit of $\hat{\eta}$ in (6).

The integrand in (14) has poles on the m-axis where the dispersion relation is satisfied, i.e. where $B = 0$. The asymptotic approximation for I can be obtained by standard contour-integral methods. The height z is assumed to be above the source, so that the radiation condition selects upgoing wave groups. This gives $I \approx 2\pi F\, e^{imz}/(\partial B/\partial m)$, where m is now a function of k,l through the dispersion relation. Thus

$$\hat{\eta}(k,l,z) = 2\pi \frac{F}{\partial B/\partial m} e^{imz}. \tag{15}$$

Eq. (15) represents the ray solution in normal-mode k,l,z coordinates. Note the difference, in terms of singularities, between it and the ray solution in spatial coordinates, i.e. the stationary-phase solution (10). The stationary-phase solution is singular where $r = 0$ and where the Gaussian curvature $\kappa = 0$. These singularities correspond to focal points of the ray paths. At $r = 0$, all rays intersect in a single focal point, as indicated by (12). When $\kappa = 0$ the rays also focus, but over the surface of a cone, as noted by Lighthill [5].

The stationary-phase singularities are not present in (15). For example, the rays that focus on the surface of a cone in spatial coordinates have different k,l values and are therefore spread out in the normal-mode coordinates. There is a singularity in (15) when $\partial B/\partial m = 0$. This corresponds to a buoyancy-frequency turning point, which we discuss in the final section.

Next we determine B and F for our specific problem. A source in a uniformly stratified medium (with a constant buoyancy frequency N) generates internal waves according to the equation (cf. (2.8) of [7], (327) of [5])

$$D^2\partial_z^2\eta + (\partial_t^2 + N^2)\nabla_h^2\eta = D\partial_z q. \tag{16}$$

The horizontal Laplacian is $\nabla_h^2 = \partial_x^2 + \partial_y^2$, and for fluid velocity $\mathbf{u}$ the source strength is $q = \nabla\cdot\mathbf{u}$. This represents the rate of volume outflow from the source per unit volume ([3], [5]). Comparing (16) with (8) we find that

$$B = \hat{\omega}^2 m^2 - (N^2 - \hat{\omega}^2)(k^2 + l^2). \tag{17}$$

The value of F is given by the Fourier transform of the spatial part of the forcing function $D\partial_z q_s$ in (16), where q_s excludes any oscillating factor $\exp(-i\sigma t)$. Thus

$$F = \hat{\omega}\, m\, Q_s, \tag{18}$$

in which Q_s represents the Fourier transform of q_s.

The procedure is then as follows. We use the above expressions for B and F to evaluate $\hat{\eta}(k,l,z)$ in (15) at a reference height z_0 just above the source. We then use that result as $\hat{\eta}_0$ in (6) and we use ray theory to work out m and $\hat{\omega}$ at the height where we desire the solution. Then we approximate the Fourier integral (7) by a discrete Fourier transform.

An example

We examine wave radiation from a sphere of radius a undergoing small oscillations of amplitude $h \ll a$. This can be modeled by ([3])

$$q(\mathbf{r},t) = -\frac{3}{2}\left[U\frac{x}{a} + h\sigma\cos(\sigma t)\frac{z}{a}\right]\delta(r-a) \tag{19}$$

where δ is the Dirac delta function. The Fourier transform of q is

$$Q(\mathbf{k},t) = -6i\pi a^3 \left[Uk + h\sigma\cos(\sigma t)m\right] \frac{j_1(Ka)}{Ka}, \qquad (20)$$

where $K = |\mathbf{k}|$, and $j_1(z) = (\sin z)/z^2 - (\cos z)/z$ is the spherical Bessel function of order 1.

There are two contributions to q, one of zero frequency and one of frequency σ, which we can treat in our theory by linear superposition. As noted by Dupont and Voisin [3], the zero-frequency component corresponds to lee waves (generated by flow relative to the obstacle) and the terms with σ corresponds to waves generated by the vertical oscillation of the obstacle.

We first consider a uniform medium in which a non-oscillating sphere moves at constant speed in the negative x-direction. We present results in the reference frame fixed to the sphere, in which the mean flow is U_0, a positive constant. Voisin [9], eq. (5.9), has derived the stationary-phase solution for a uniformly moving dipole source using a Green's-function technique. Modified for the source function q used here, Voisin's solution becomes

$$\eta(\mathbf{r},t) \sim H(x)\frac{NQ}{2\pi U_0^2}\frac{xz}{R}\frac{(x^2y^2+R^4)^{1/2}}{x^2+R^2} \cos\left[\frac{N|z|}{UR}(x^2+R^2)^{1/2}\right], \qquad (21)$$

where H is the Heaviside function, $R = (x^2+y^2)^{1/2}$, and Q is given by (20) with $\sigma = 0$. The wavenumber dependence in Q is related to position by the stationary-phase condition (12).

Figure 1 shows Voisin's solution for η/a, where a is the sphere radius. All axes in this paper are labelled in distance normalized by a. The Froude number is $Fr = U_0/Na = 1$, and $z = 5a$. Figure 2 shows Maslov's solution for the same problem, in close agreement with Voisin's solution. Maslov's solution is computed with a 256 by 256 Fourier-series approximation to (5). This gives a domain size of about $80a$ in y and about $200a$ in x, beyond which the solution wraps around periodically.

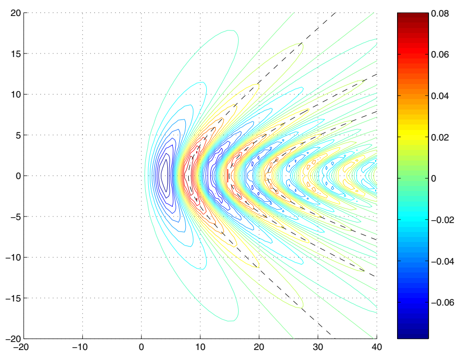

Figure 1: Voisin's stationary-phase solution (21) for η/a at $z = 5a$. The axes are the horizontal positions x/a and y/a, with x/a horizontal on the page. Dashed lines indicate the positions of three lines of constant phase.

Voisin's theory is limited to a uniform medium, but Maslov's solution (5) has been derived to allow depth-dependent mean velocity and mean buoyancy. We continue to assume constant N but now add a cross current with constant shear. All other parameters are the same as in Figures 1 and 2. The cross current has the form $V(z) = Rz$, where R is a constant. We choose

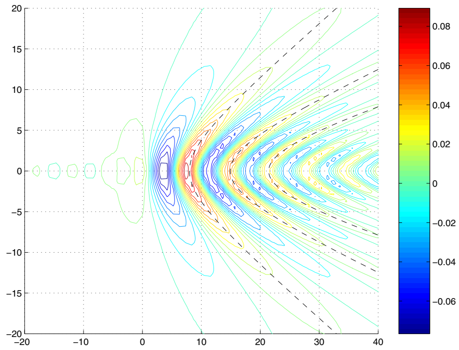

Figure 2: Maslov's solution corresponding to Figure 1.

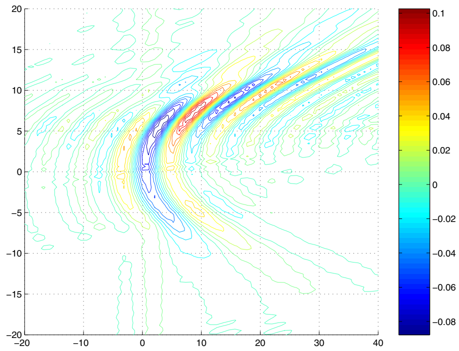

Figure 3: Maslov's solution for the same parameters as in Figure 2, but in the presence of cross shear.

R such that $N/R = 25$. The result of Maslov's method is pictured in Figure 3. Rays that in Figure 2 contribute to the southern wing ($y < 0$) of the lee waves are now refracted toward buoyancy-frequency turning points, where $\hat{\omega}$ approaches N and the waves are reflected vertically. Different rays have different turning-point heights, but much of the lee-wave energy in this southern wing gets reflected by the height plotted in Figure 3. After reflection, the corresponding ray component is deleted from our calculation. (That is, we compute only contributions from the upgoing waves.) Rays that in Figure 2 make up the northern wing ($y > 0$) of the lee waves are now refracted toward critical layers, where $\hat{\omega}$ becomes vanishingly small. Each ray approaches its own critical layer height asymptotically. The critical-layer behaviour is very similar to that occurring in the mountain-wave study of [8].

Discussion

The point of using Maslov's method instead of the traditional ray approach is that Maslov's method has a better range of validity. In our application of Maslov's method, the ray approximation is used for (3). It is well known that the ray solution to (3) is valid when the slowly-varying criterion

$$\left|\frac{1}{m^2}\frac{\partial m}{\partial z}\right| << 1 \qquad (22)$$

is satisfied. Consider for simplicity the case of constant N. It can be shown from the dispersion relation that

$$\left|\frac{1}{m^2}\frac{\partial m}{\partial z}\right| = \left[\frac{|kU_z + lV_z|}{(k^2+l^2)^{1/2}N}\right]\left[\frac{1}{(1-\hat{\omega}^2/N^2)^{3/2}}\right]. \qquad (23)$$

The term in the first square brackets on the right-hand-side of (23) is small if the Richardson number is large. For large

Richardson number, the ray solution to (3) fails only where $\hat{\omega}$ is close to N, i.e. near a buoyancy-frequency turning point. This is a caustic in the transform domain.

Despite this failure, Maslov's method has advantages over traditional ray methods, which also break down at caustics. One reason is that the $\hat{\omega} = N$ caustic in the transform domain is easy to correct because the governing equation (3) is one-dimensional. It can be shown that as the ray approaches the caustic, the ray solution to (3) begins to fail when $|m^{-2}dm/dz| \approx 1$. This condition can be monitored in a ray calculation and can be used to determine where the ray solution to (3) should be replaced by an Airy-function solution to (3), which is straightforward to calculate in this case.

In the spatial domain, the caustic-correction procedure is significantly more difficult. The caustics can occur at any value of $\hat{\omega}$, and they can have a range of forms, from the simplest Airy-function caustic to caustics with cusps and other characteristics that render the Airy-function representation inadequate. (See also [1] for a caustic that is not correctable with an Airy function.) Even for the simple Airy-function caustic, the solution near the caustic depends on the curvature of the caustic surface, which must be determined.

There are other problems with the traditional ray method. For example, Lighthill's stationary-phase solution (10) diverges for an oscillating source that does not move relative to the fluid. This is a case where the Gaussian curvature of the wavenumber surface vanishes everywhere. As Lighthill explains, the singularity is removed when the source is put into motion. But how fast do we have to move the source for the stationary-phase result to become valid? More generally, how do we know when an increase in wave amplitudes predicted by a ray calculation is a genuine prediction of linear theory, or a symptom of the failure of the slowly varying approximation? There does not appear to be a general criterion for assessing where ray theory breaks down that is as useful and reliable as the one noted above in relation to the one-dimensional equation (3), involving $m^{-2}dm/dz$.

Maslov's method is based on the simple idea that the normal-mode equation (here, (3)) does not have to be solved exactly. Using a ray approximation instead is practical and overcomes a number of limitations of traditional ray methods. We have also obtained results (to be reported elsewhere) for an oscillating source, as in [3] but including mean shear. Further work is needed to determine how effectively Maslov's method can model vertically trapped modes, Coriolis effects, and horizontally varying media. These are areas for future research.

Acknowledgements

Funding was provided by NASA through the Atmospheric Chemistry Modeling and Analysis Program (ACMAP grant L68786D) and the Upper Atmospheric Research Satellite Guest Investigator Program (grant NAS5-98045). Additional funding was provided by the Office of Naval Research through the Turbulence and Wakes Program (Dr. L. Patrick Purtell, program manager) under contract number N00014-99-C-0270.

References

[1] Broutman, D., Rottman, J.W. and Eckermann, S.D., Maslov's method for stationary hydrostatic mountain waves, to appear in *Q. J. Roy. Met. Soc.*, 2001.

[2] Brown, M.G., The Maslov integral representation of slowly varying dispersive wavetrains in inhomogeneous moving media, *Wave Motion.*, **32**, 2001, 247–266.

[3] Dupont, P. and Voisin, B., Internal waves generated by a translating and oscillating sphere, *Dyn. Atmos. Oceans*, **23**, 1996, 289–298.

[4] Gill, A.E., *Atmosphere-Ocean Dynamics*, Academic Press, 1982.

[5] Lighthill, M.J., *Waves in Fluids*, Cambridge University Press, 1978.

[6] Maslov, V.P. and M.V. Fedoriuk, Semi-Classical Approximation in Quantum Mechanics, D. Reidel. 1981.

[7] Miles, J.W., Internal waves generated by a horizontally moving source. *Geophys. Astrophys. Fluid Dyn.*, **2**, 1971, 63–87.

[8] Shutts, G.J., Stationary gravity-wave structure in flows with directional wind shear. *Q. J. Roy. Met. Soc.*, **124**, 1998, 1421–1442.

[9] Voisin, B., Internal wave generation in uniformly stratified fluids. Part 2. Moving point sources, *J. Fluid Mech.*, **261**, 1994, 333–374.