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Görtler vortices in curved mixing layers and their effect on the inherent instabilities

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Abstract

Curved mixing layers are known to be inherently unstable to short wavelength travelling waves (due to the layer's inflectional nature) and longitudinal vortex structures (due to the change in circulation); these are akin to Görtler vortices. We shall discuss the latter through their linear evolution and subsequent nonlinear growth. Depending on the size of the instability and other parameters this may cause the flow to breakdown. This manifests itself as large jets in the neighbourhood of the centre line of the mixing layer. Prior to the actual breakdown, the flow becomes inflectional in the spanwise coordinate and has additional structure in the normal direction. One is compelled to solve the Rayleigh stability equation with variation in the underlying state in more than one variable; this requires a sophisticated numerical method to solve the resulting partial differential eigenvalue problem. The presence of the vortices and their effect on the intrinsic travelling waves are discussed, with a view to using their existence to control these waves.

Introduction

It is well known that the presence of curvature promotes the evolution of longitudinal vortices. Within mixing layers these instabilities have been observed experimentally by, for instance, Plesniak, Mehta & Johnson, [8] and studied in a host of theoretical articles cited herein. These vortices evolve over a similar scale to the mixing layer and have spanwise wavelength commensurate with the layer's thickness, which renders the governing equations parabolic in nature.

In Hall [2] the linear evolution of these modes within the context of boundary layers was solved. It was shown to be crucial to include the coincident evolution of the layer; this led to an inevitable sensitivity to initial conditions, which is clearly demonstrated in experimental work, [10]. It is probable that these vortices will grow to such amplitudes that it is necessary to consider nonlinear effects [2]. In the context of boundary layers the calculations breakdown and the condition of parabolicity is violated; due to a reversed flow. However, prior to this location the flow has become inflectional in both the normal and spanwise coordinates. It is therefore likely to be prone to fast growing travelling waves, [3].

Within the context of mixing layers (both incompressible and compressible) a similar process occurs and it is our intention to discuss the corresponding structures. The linear evolution of the modes were discussed [6, 9] and the nonlinear fate of the modes within an asymptotic forum was given in [9, 7]. There are several interesting results which can be derived analytically which rely on the consideration of the circulation criterion, [9].

The equation governing the secondary instability of the composite flow (that is containing the underlying profile and the amendments due to the vortices) is merely the two-dimensional counterpart of the Rayleigh equation. It can be solved using a mixture of finite differences and pseudo-spectral techniques. The required eigenvalues are obtained using both local [5] and global methods: the latter are used merely as a precursor to the local methods, but their computational expense make their use inappropriate for the determination of high resolution solutions.

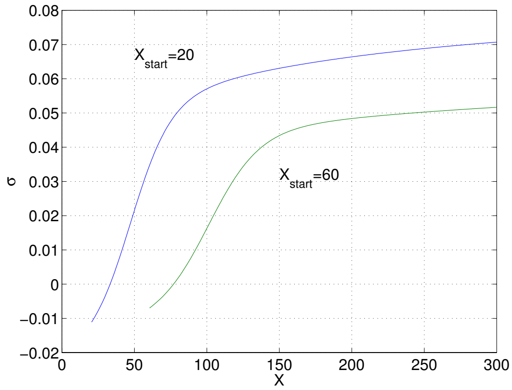

Figure 1: This depicts the growth rate as a vortex with $a = 0.08$ evolves downstream, for two starting locations.

Linear and Nonlinear vortices

The physical problems considered herein can be categorised using a few select parameters: Reynolds number, Görtler number, the spanwise wavelength of the vortex (and its initial amplitude), the short streamwise wavelength of the secondary instabilities, [3, 6]. After the equations have been non-dimensionalised and the requisite limits have been taken the equations governing the linear evolution of the vortices can be determined. These are similar to those governing the motion in a narrow layer (that is the rôle of streamwise diffusion has been removed) but with the removal of the streamwise pressure derivative. This means that the equations are parabolic and consequently amenable to a marching solution (for which we use a Crank-Nicolson scheme). The equations are manipulated to eradicate the vortex pressure and spanwise velocity component. This leads to a system of two coupled partial differential equations (of fourth and second order).

As the vortex evolves downstream its growth is measured by calculating an associated energy. Results are shown in figure 1 for a representative case. Note that the vortex initially decays and subsequently grows. We conjecture for all but the smallest initial disturbances, nonlinear effects will play a rôle; in fact this will become evident during our discussion of the nonlinear calculations.

In figure 2 we show two neutral curves in terms of the local spanwise wave number and Görtler number (which change as a result of the spreading of the mixing layer).

The two curves correspond to different initial conditions [6]. The right hand branches of the two curves asymptote together; which corresponds to the calculations far downstream. This provides further evidence that the inclusion of the evolution of the modes is crucial.

The derivation of the nonlinear equations relevant to the vortices for the mixing layer is identical to that for the boundary layer, [2]. However in the consideration of the boundary conditions we find a subtle difference. In the boundary-layer case the necessity for the plate to be impermeable yields a condition on the normal velocity there; this is obviously not present in the mixing layer. Consequently we need to find an additional condition (together the conditions that the streamwise velocity at the extrema of the layer must match with the freestream values). We elect to use the Ting condition which balances pressures across the layer. Not only does this complete the system for the underlying mixing-layer profile but it also gives us a condition on the mean-flow correction.

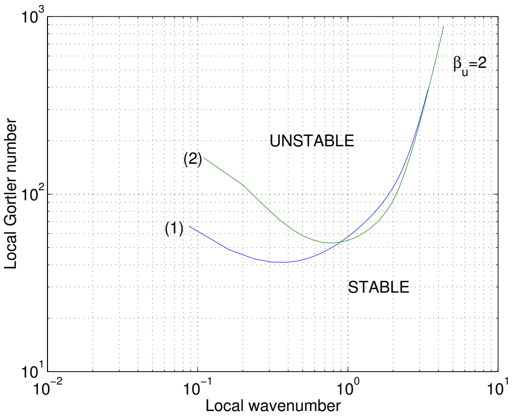

Figure 2: Neutral curves associated with two different initial conditions for the ratio of stream speeds equal to 2, with curvatures proportional to $\sqrt{x}$.

Each of the components associated with the vortex is expanded as a Fourier series of period $2\pi/a$ (where a is the spanwise wavenumber of the fundamental mode). These series expansions naturally require truncation and the results presented herein considered the first 16 harmonics; this value has been demonstrated to be satisfactory. The nonlinear terms require the convolution of the two series and in general this is done in the Fourier space, rather than reverting to physical coordinate. We pause here to comment that the linear mode form is applied at $x = 20$ and the nonlinear terms are initiated at $x = 40$; this allows the initial form of the mode to be integrated for a distance downstream. It is realised that in reality a receptivity problem should be solved, which would provide us with an initial spectrum. The ultimate downstream structure is influenced by these decisions but is quite robust in overall form. The location at which the flow experiences an explosive growth of the fundamental may change by a few percent but the resulting flow will still, nevertheless, become highly inflectional.

In figure 3 we show the energy which is defined in terms of the streamwise velocity component, associated with the first four modes. The fundamental mode's energy initially decays, as it will in the absence of the nonlinear terms and subsequently starts to grow. Significantly the mean-flow correction grows to have almost the same

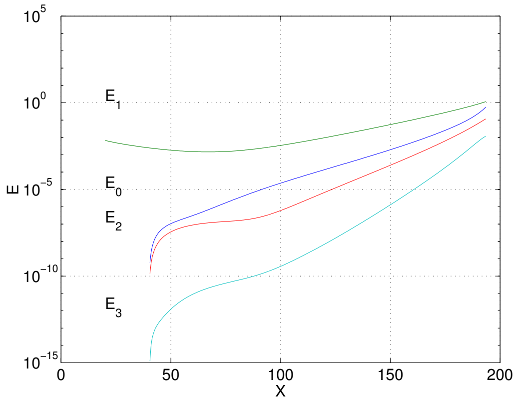

Figure 3: This depicts the energy associated with the fundamental, second and third harmonics, together with that of the mean-flow correction.

amplitude as the undisturbed flow. Notice that the energy associated with the third harmonic remains, in some sense, small compared to that of the fundamental. The initial amplitude shown here is 5%. There is an explosive growth at approximately $x = 195$. This phenomenon occurs for all amplitudes above a threshold amplitude: this may well be delayed but will almost always occur provided the parameters are conducive to linear growth. In figure 4 we show the growth rate associated with five runs with varying initial amplitudes. It is tempting to say that at the point at which there is this explosive growth the flow will undergo a transition. We also show the flow profile immediately before breakdown in figure 5. However it is crucial that we consider temporal instabilities in addition to the spatial ones considered above. In fact it is our conjecture that the flow will never attain this state, as it will experience transition due to temporal instabilities, Hall & Horseman [3]. Notice as the modes evolve they grow and overcome the basic flow, the location of the break down in the calculation is shown in figure 4 (which coincides with the spike in the growth rate of the fundamental).

It is worth noting that although few runs are shown here, our experience is that this structure pervades other parameter régimes. This may not be the case if this calculation were to be extended to a fully compressible régime. In that case many different issues come into play. For instance, the presence of a thermal gradient may cause modes to become excited. These have been shown to be similar in character but can lead to an inhibition of the underlying inviscid instabilities, [7]. In fact by making a Boussinesq approximation one can start to understand the mechanisms which are competing in situations with curvature and thermal gradients. In Watson & Otto [11] discussion is given concerning the way in which asymptotic and numerical treatment of the problem can identify critical levels of interaction, that is where stabilising curvature can dominate over unstable stratification.

Rayleigh modes

Mixing layers are necessarily inflectional and are consequently prone to inviscid travelling waves [4]. In the absence of these vortex structures we note that the travelling waves satisfies Rayleigh's equation (which can be expressed in terms of the normal velocity or the pressure). However the presence of spanwise variations means that the travelling waves now satisfy a partial differential

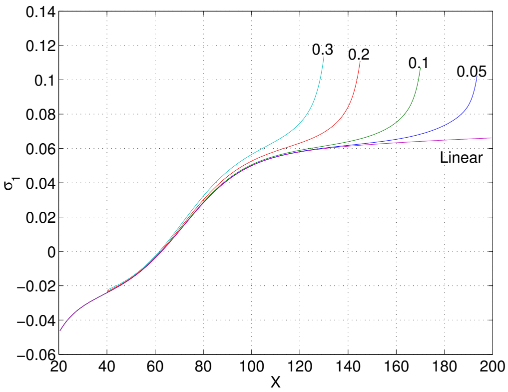

Figure 4: Growth rates for the fundamental for various initial amplitudes, with the linear results.

eigenvalue problem, which can now only be expressed in terms of the wave's pressure component:

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\alpha^2\right)\tilde{p}-\frac{2}{\bar{u}-c}\left(\frac{\partial \bar{u}}{\partial y}\frac{\partial \tilde{p}}{\partial y}+\frac{\partial \bar{u}}{\partial z}\frac{\partial \tilde{p}}{\partial z}\right)=0.$$

This needs to be solved in conjunction with the conditions that the pressure has the same periodicity as the fundamental mode in the spanwise coordinate and that at the layer's extrema it tends to zero. In this equation the basic flow is comprised of the underlying mixing-layer profile and the vortex state. In this article we shall calculate the complex phase speed c which corresponds to a value of the streamwise wavenumber α. We note that this is not necessarily unique. The imaginary part of the phase speed is representative of the temporal growth rate of the modes. We shall now discuss the methods used to solve this equation.

Numerical Methods

In order to solve the eigenvalue problem at hand we exploit a standard centred difference approach on the normal derivatives on a stretched grid. In general this is a subset of the points used for the marching calculations, this eradicates the need for interpolation between the grids. The Rayleigh equation is discretised in the spanwise coordinate, however the spanwise derivatives are calculated using the Fourier transforms. These differential operators are constructed by producing the composition of the following operations: transform from physical to Fourier space, differentiate in the Fourier space and finally transform from Fourier space to physical coordinates. We exploit this method since we are aware that the following numerical approach means that the blocks will become full and no sparsity can be exploited.

We employ both local and global eigenvalue search techniques. In order to solve the global eigenvalue problem we can exploit a power method which is intrinsic to MATLAB. This allows us to determine the form of various eigenfunctions for a specific value of the wave's streamwise wave number. The expense of this calculation is restrictive and consequently we use the algorithm presented in Otto & Denier [5] (which details a local eigenvalue solution technique). This overcomes the problem of renormalisation intrinsic to these large eigenvalue problems. Without normalisation the problem is homogeneous and only the trivial solution is returned. The above discretisation results in a matrix system, $\mathbf{A}\mathbf{x}=\mathbf{0}$, where $\mathbf{A}$ is a square block diagonal matrix (this is generally block pentadiagonal with full blocks) and the vector $\mathbf{x}$ contains the pressure field. We note that in order for there to exist a non-trivial solution $\det(\mathbf{A})$ must be zero and this determines the values of c. The calculation of this determinant is far from trivial. Here we elect to remove one row from the matrix and replace it with the condition that $\mathbf{x}_j=1$ (where we are free to choose j) the resulting system can be solved and the dot product of the extracted row and the solution can be used as an error. This is driven to zero using a secant method. The choice of which row to remove is not fixed and is moved using an algorithm which relies on the previous iterate [1].

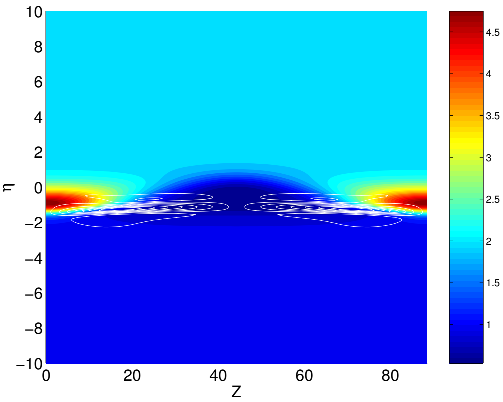

Figure 5: Showing the colour contours of the streamwise velocity, noting the large jetting super-velocities. Also shown are contours of streamwise vorticity.

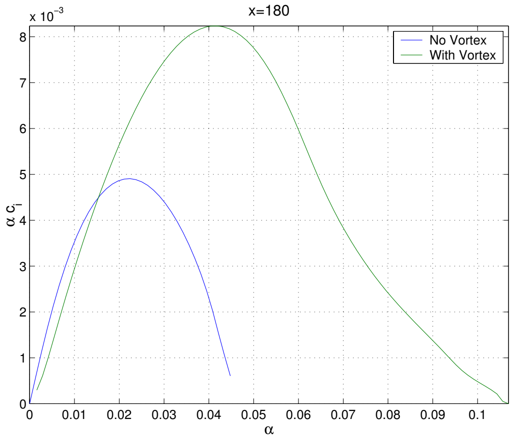

Figure 6: Growth rates of the inviscid instabilities associated with the mixing layer, at $x=180$ and an initial amplitude of 5%.

In figure 5 we show a flow field almost at the point of breakdown of the calculation. The figure shows two distinct facets of the profile: the streamwise velocity component and the streamwise vorticity. We note that the streamwise velocity near the centre line increases dramatically at the spanwise location corresponding to the maximum of the vortex and unsurprisingly, decreases in the centre of the period. These jets have an involved spanwise structure. By the inclusion of the streamwise vorticity contours we hope to show the trajectory of particles within the flow and show the dominant vortical structure.

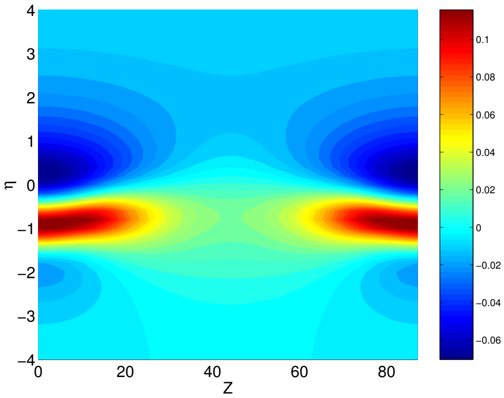

Figure 7: This shows a contour plot of the pressure wave associated with the Rayleigh mode.

In figure 6 we show the growth rate associated with the profile in the presence and absence of the vortex state. Notice that due to the spreading of the mixing layer the growth rate of the conventional inviscid instability decreases as the flow evolves downstream. Also shown is the inviscid instability, whose existence is inherently linked to the spanwise structure. This has a larger growth rate and persists for more values of the scaled streamwise wave number of the waves. In figure 7 we show the contours of the wave's pressure component (the imaginary part, although due to the normalisation this choice is arbitrary) which clearly indicates the variation in the spanwise direction. In fact information can be gleaned from the Fourier decomposition of the modes.

It is noted that at the outset the dominant inviscid instabilities are those associated with the inflection of the basic flow. This remains the case until a reasonable distance downstream, however at some point the modes associated with the vortices start to dominate. It is interesting to note that this provides a further mode of instability to a situation which is already absolutely unstable.

Conclusions

In this article we have discussed the evolution of Görtler vortices within the context of curved mixing layers. In the linear régime the modes' fate is inextricably linked to the inception of the mode, both in terms of its location and its functional form. However, far downstream the modes are found to conform to a common structure, a fact which has been previously exploited by theoretical investigations. As the modes grow eventually nonlinear effects will come into play. Within the context of boundary layers this has been shown to lead to a breakdown of the solution; a similar fate occurs here. It appears that large super-velocities occur (of the order of twice the maximum flow velocity) and the flow in the centre drops below the speed of the slower stream. It is not clear whether the flow will ever attain this state, since it is likely that temporal instabilities will come into play.

We have shown that as the vortices evolve downstream inviscid instabilities become dominated by a mode associated with the spanwise structure rather than being shadows of the inherent modes. In order to determine the structure of these new modes it is found to be necessary to include the effects of the mean-flow correction, the fundamental and the second harmonic. Despite the amplitude of the first and third of these being smaller, their presence is crucial (with the second harmonic being perhaps the most important). This has ramifications when considering possible mechanisms for vortex-wave interaction, Hall & Smith [4]. This mechanism provides a route, whereby fast growing instabilities can have a leading-order effect on the mean flow, without becoming nonlinear in character themselves.

As mentioned previously the inclusion of thermal gradients in these problems can have a significant effect. This will manifest itself not only in the modification of the intrinsic instability mechanisms but in the excitation of modes even in convex situations [9]. This all means that the level of curvature and thermal stratification can potentially be used to tune the instabilities, and consequently enhance or inhibit transition.

Acknowledgments

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