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Three Dimensional Transition in a Bickley jet: Comparison of Theory with DNS

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Abstract

We consider interactions between varicose and sinuous oblique disturbances in the Bickley jet, using both nonlinear stability theory (in its nonlinear critical layer form) and direct numerical simulation using a spectral method. Nonlinear stability theory indicates that a (nonlinear) interaction between the modes should occur, and our simulations would seem to support this.

Introduction

For plane wakes and jets, it is well known that there may be two different types of neutral modes with critical layers centred on the inflection points, viz. the sinuous and varicose modes. The plane (Bickley) jet, which has a $\mathrm{sech}^2 y$ velocity profile has been used by numerous authors to provide a good approximation to such a wake behind a bluff body, is somewhat special in that the varicose and sinuous modes have neutral wavenumbers of 1 and 2 respectively, so that the former is the subharmonic of the latter. Several studies have explored the possibility of an interaction between these two modes. For two-dimensional disturbances, Kelly [5] used Stuart-Watson type nonlinear stability theory to investigate interactions of this type; however, he found that there was no modal interaction of the type assumed. Later, Leib and Goldstein[7] re-examined the problem for purely two-dimensional disturbances using a nonlinear-nonequilibrium critical layer, and they found that there was indeed an interaction between the modes. Mallier[8, 9] studied three-dimensional disturbances, since Goldstein[3, 2] had earlier shown that a pair of oblique waves superimposed on a shear layer could interact nonlinearly to give rise to extremely rapid growth, and also[4] that when the oblique waves were inclined at $\pm 60^\circ$, an additional interaction could take place between the oblique waves and a plane wave, so that the growth was faster still; this last mechanism is known as a "resonant triad". Mallier[8] explored the possibility of an interaction between a pair of resonant triads in the Bickley jet, with one triad consisting of a plane sinuous mode together with a pair of oblique sinuous modes inclined at $\pm 60^\circ$ and the other triad consisting of a plane varicose mode together with a pair of oblique varicose modes also inclined at $\pm 60^\circ$; the motivation behind this was to see if the triads could interact so that the growth was more rapid than if only a single triad were present. Mallier found that interactions could occur, and his study essentially covered three stages: a linear stage when the amplitudes of the disturbances were very small, the "parametric resonance" stage, and the so-called "fully-coupled" stage[4, 18, 10]. The amplitude equations presented were of course for the third (fully-coupled) stage, but the two earlier stages could be recovered from these equations by rescaling the amplitudes, as discussed in[4]. The study of the fully-coupled stage was a little restrictive in that it was necessary to assume that, in that stage, the varicose oblique modes were larger any of the other waves present which is at odds with the linear theory which says that the linear growth rates of the sinuous modes are larger than those of the varicose modes. This was because in the earlier parametric resonance stage, when it was assumed that all of the waves were of the same order of magnitude, it had been found that the varicose oblique waves underwent very rapid growth while the plane waves and the sinuous oblique waves continued to grow exponentially in a linear fashion. In addition, the coupling in the equations in [8] was a little unusual in that the sinuous triad did not affect the varicose triad, and therefore equations for the varicose triad were simply those for a single resonant triad[4, 18, 10]. However, the sinuous triad was strongly affected by presence of the varicose triad, and furthermore, if the varicose triad was absent the nonlinear terms in the equations for the sinuous triad vanished, leaving only linear equations for those modes. Mallier[9] later studied the case of two pairs of oblique waves superimposed on the Bickley jet at the same angle, $\pm\theta$: one pair was varicose, the other sinuous. Once again, an interaction was found to occur between the modes. In both studies, [8, 9] the end result was a set of highly nonlinear coupled (Hickernell-type) integro-differential evolution equations, the solutions to which had a finite-time singularity. These equations involved a nonlinear kernel, with the nonlinearity being cubic in[9] and quartic in[8]. Both studies[8, 9] used nonlinear critical layer theory and followed the approach taken earlier by Goldstein[3, 4] for a flow with a single critical layer and a single unstable mode.

The reason these interactions between the varicose and sinuous modes are considered important is that it is possible they can cause extremely rapid nonlinear growth. Unfortunately, at the time the studies mentioned above were performed, there was little if any experimental or numerical evidence to corroborate our analysis, and the situation remains the same today. Some experiments have hinted at interactions, but have not explored it further. Wygnanski *et al.* [20] conducted careful experiments on small deficit (turbulent) wakes and found that the development of some aspects of the flow was dependent on initial conditions, which they attributed to interactions between the varicose and sinuous modes, and other experiments (e.g.[11, 12]) have also suggested that these interactions may take place. We should also mention that very rapid amplification of three-dimensional disturbances has indeed been observed in plane wakes in both experiments (e.g. [1, 16, 17]) and numerical simulations[15], but it is unclear (at least to the present author) how much of that growth is attributable to the Goldstein mechanisms for three-dimensional instability and how much is due to an interaction between the varicose and sinuous modes.

In an attempt to remedy what we perceive as a lack of numerical verification of the theoretical results, we present here some preliminary results from direct numerical simulations of the Bickley jet and compare those results to the predictions of our earlier asymptotic analysis. The outline of the rest of the paper is as follows. In the next section, we review the theory and resulting amplitude equations. After that, we give the details of our numerical method and present the results of our simulations. Finally, we make some concluding remarks.

Review of Theory

In [8, 9], we considered the stability of the Bickley jet $\underline{u}_0 = (\text{sech}^2 y, 0, 0)$ to three-dimensional disturbances, either in the form of two pair of oblique waves,

$$\begin{aligned} & 2\varepsilon A_{11}\underline{\tilde{u}}_{11}e^{i\alpha\cos\theta(x-ct)}\cos[\alpha\sin\theta z] + c.c. \\ + \quad & 2\varepsilon A_{22}\underline{\tilde{u}}_{22}e^{2i\alpha\cos\theta(x-ct)}\cos[2\alpha\sin\theta z] + c.c. \quad (1)\end{aligned}$$

or in the form of a pair of resonant triads, consisting of a two pair of oblique waves at ±60 together with two plane waves,

$$\begin{aligned} & 2\varepsilon A_{11}\underline{\tilde{u}}_{11}e^{i\alpha(x-ct)/2}\cos\left[\alpha\sqrt{3}z/2\right] + c.c. \\ + \quad & 2\varepsilon A_{22}\underline{\tilde{u}}_{22}e^{2i\alpha(x-ct)/2}\cos\left[\alpha\sqrt{3}z\right] + c.c. \\ + \quad & \varepsilon^{4/3}A_{20}\underline{\tilde{u}}_{20}e^{i\alpha(x-ct)} + c.c. \\ + \quad & \varepsilon^{4/3}A_{40}\underline{\tilde{u}}_{40}e^{2i\alpha(x-ct)} + c.c.. \quad (2)\end{aligned}$$

From the linear theory, we know that the Bickley jet has two inflection points at $y = \pm\text{arccosh}\sqrt{3/2}$ where $u_0 = 2/3$, and also two neutral modes with $c = 2/3$ which have critical layers centered on the inflection points: a varicose mode with $\alpha = 1$ and $\hat{v} = \text{sech}\, y \tanh y$ [13] and a sinuous mode with $\alpha = 2$ and $\hat{v} = \text{sech}^2 y$ [14]. In the above disturbances, we have either two amplitudes (for the pairs of oblique waves) or four amplitudes (for the resonant triad) and the objective of [8, 9] was to derive amplitude equations for those amplitudes, and examine how they affected each other. In the analysis, it was assumed that the modes were periodic in time and spatially growing, with the wavenumber α taking the neutral value of 1 and phase velocity being perturbed slightly from neutral $c = 2/3 - \mu c_1$. This perturbation from neutral led to a slow time scale $T = \mu t$ on which the amplitudes evolved. For extremely small disturbances, it was found that the growth was linear, with $A_{11}(T) = A_{11}^{(0)}e^{\sigma_{11}X}$ and similar expressions for the remaining modes, but for larger disturbances the evolution was nonlinear; the evolution first became nonlinear when $\varepsilon = \mu^3$. For pairs of oblique waves, it was found that the amplitude equations were (Hickernell-type) integrodifferential equations with a cubic nonlinearity of the form

$$\begin{aligned} & A_{11}' + \sigma_{11}A_{11} \quad (3) \\ = \quad & \int_0^\infty\int_0^\infty \kappa_1^{(a)}A_{11}^*(T-2\tau_0-\tau_1) \\ \times \quad & A_{11}(T-\tau_0-\tau_1)A_{11}(T-\tau_0)\,d\tau_0 d\tau_1 \\ + \quad & \int_0^\infty\int_0^\infty \kappa_1^{(b)}A_{22}^*(T-3\tau_0-\tau_1) \\ \times \quad & A_{22}(T-2\tau_0-\tau_1)A_{11}(T-2\tau_0)\,d\tau_0 d\tau_1 \\ + \quad & \int_0^\infty\int_0^\infty \kappa_1^{(c)}A_{22}^*(T-\tau_0-2\tau_1) \\ \times \quad & A_{22}(T-\tau_1)A_{11}(T-2\tau_0-2\tau_1)\,d\tau_0 d\tau_1 \\ + \quad & \int_0^\infty\int_0^\infty \kappa_1^{(d)}A_{22}^*(T-3\tau_0-2\tau_1) \\ \times \quad & A_{22}(T-2\tau_0-\tau_1)A_{11}(T-2\tau_0-2\tau_1)\,d\tau_0 d\tau_1\end{aligned}$$

and

$$\begin{aligned} & A_{22}' + \sigma_{22}A_{22} \quad (4) \\ = \quad & \int_0^\infty\int_0^\infty \kappa_2^{(a)}A_{22}^*(T-2\tau_0-\tau_1) \\ \times \quad & A_{22}(T-\tau_0-\tau_1)A_{22}(T-\tau_0)\,d\tau_0 d\tau_1 \\ + \quad & \int_0^\infty\int_0^\infty \kappa_2^{(b)}A_{11}^*(T-3\tau_0-\tau_1) \\ \times \quad & A_{11}(T-\tau_0-\tau_1)A_{22}(T-\tau_0)\,d\tau_0 d\tau_1 \\ + \quad & \int_0^\infty\int_0^\infty \kappa_2^{(c)}A_{11}^*(T-3\tau_0-2\tau_1) \\ \times \quad & A_{11}(T-\tau_1)A_{22}(T-\tau_0-\tau_1)\,d\tau_0 d\tau_1.\end{aligned}$$

For the resonant triad, the equations were of a similar form but with a quartic rather than a cubic nonlinearity. In [9], we solved these equations numerically, using Goldstein's numerical scheme [3], and it was found that, as with similar problems, the evolution of the disturbances went through 3 stages: initially, the disturbances grew linearly, until a second finite-amplitude nonlinear stage was reached, and eventually, the oblique waves experienced explosive growth. Goldstein [3] showed that his equations had a singularity after a finite time T_s (or at a finite distance downstream), and was able to fit a structure to it $A \sim a_0(T_s - T)^{-3-i\psi}$. This same structure applies to both modes for the Bickley jet, and is shown in cartoon form in Fig. 1. Although our study did not include viscous effects, other studies for related problems [6, 19] have shown that weak viscosity can delay the onset of this finite time singularity but not eliminate it, so that our results are still meaningful at high Reynolds numbers. The origins of this finite-time singularity are still not entirely clear, but it appears to be connected to the breakdown of the theory and the onset of a new, still more nonlinear stage governed by the full Euler equations.

Numerical Simulations

We now turn to our numerical simulations, in which we have employed a standard spectral (Fourier) method. In

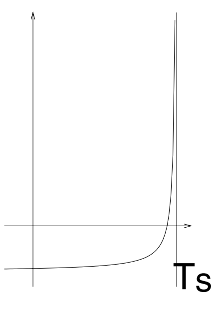

Figure 1: Cartoon of the finite-time singularity.

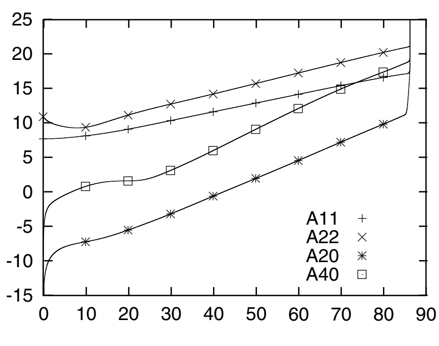

Figure 2: 3D Run1

our computations, we decomposed the velocity into a mean flow and a perturbation, $\underline{u} = (u_0(y), 0, 0) + \tilde{\underline{u}}$, and then assumed that the mean flow is independent of time and the viscosity acts only on the perturbation; in reality, this would require the presence of a body force to counteract the effect of viscosity on the mean flow. In our simulations, we took the base flow to be the jet $u_0(y) = \mathrm{sech}^2 y - 2/3$; it should be noted that the "$-2/3$" term is included so that the neutral modes in our simulation have a phase velocity of zero. Using this standard decomposition, the perturbation obeys

$$\frac{\partial \tilde{\underline{u}}}{\partial t} = -\underline{u} \bullet \nabla \underline{u} - \nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \tilde{\underline{u}}$$
$$\nabla \bullet \tilde{\underline{u}} = 0. \quad (5)$$

The approach we took was to assume that the flow was periodic in both x (streamwise) and z (spanwise), and use complex Fourier series in those directions. We truncated the y-direction, so that our domain was $-Y \leq y \leq Y$ (with $Y = 5$ in our simulations) rather than $-\infty < y < \infty$ and used sines and cosines in that direction. Free-slip conditions were applied at the domain boundary $y = \pm Y$.

We calculated the nonlinear ($\underline{u} \bullet \nabla \underline{u}$) terms in physical space and derivatives in Fourier space, and used a fast Fourier transform (FFT) to switch between real and physical space. In our computations, it was necessary to calculate the pressure, which we were able to do by using incompressibility and then inverting a Laplacian,

$$p = -\left(\nabla^2\right)^{-1} \nabla \bullet \left(\underline{u} \bullet \nabla \underline{u}\right); \quad (6)$$

this is fairly easy to do with a Fourier method. For the time-stepping, we used an explicit Adams-Bashforth scheme for the nonlinear terms,

$$\tilde{u}_1 = \tilde{u}_0 + \delta t \left(3F_0 - F_{-1}\right)/2, \quad (7)$$

while for the viscous terms, a semi-implicit Adams-Moulton scheme was used,

$$\tilde{u}_1 = \tilde{u}_0 + \delta t \left(F_1 + F_0\right)/2. \quad (8)$$

In each of our runs, the initial disturbance was the linear *inviscid* eigenvalue with wavenumber 0.525 for the varicose mode and 1.050 for the sinuous modes; this initial disturbance was obtained by solving Rayleigh's equation numerically.

In our simulations, we calculated the energy in each mode,

$$\int_{-Y}^{Y} \int_0^{2\pi/\alpha_x} \int_0^{2\pi/\alpha_z} |\underline{u}_{mn}|^2 \, dz dx dy, \quad (9)$$

and compared this to the theoretical amplitude of the disturbance. We should mention however that strictly speaking, therefore, we are not comparing like-with-like, since the two definitions differ slightly. Similarly, our simulations include weak viscosity (a Reynolds number of 1500 was used) which is necessary for numerical stability, while our theory was inviscid. However, as we discussed earlier, studies by other authors have indicated that our theory is applicable to high Reynolds number flows. In addition, we took a wavenumber of $\alpha = 0.525$ in each of our runs, which was chosen to be reasonable close to the most unstable wavenumber for both the varicose and the sinuous modes, while our theory assumed that α was very close to the neutral value of 1; we could not use a significantly larger wavenumber in the simulations because the presence of viscosity means that modes which are close to neutral on an inviscid basis would decay in simulations.

Sample 3D Runs

In Fig.s 2-5, we present several different runs for the resonant triad, with $\theta = \pi/3$. Each of the runs shown eventually blows up as we lose spectral decay and the higher harmonics become too large. Paradoxically, this is similar to the reason for the finite-time singularity in Goldstein's amplitude equation. Since the location of the blow-up is different for each of the runs presented, this by itself would indicate that an interaction between the modes is occuring. Our simulations indicate that we are able to capture the nonlinear stage that appeared in the solution to Goldstein's equation, and for the three-dimensional case it would appear that the modes do indeed exert a significant influence on each other; an example of this can be seen in the behavior of A_{11} in Fig.s 2 and 3: the initial conditions for A_{11} were the same in both runs, and the difference in the behaviors of this mode in the two runs is due entirely to nonlinear interactions between the modes. The initial conditions for the runs shown are as follows: in Run 1, A_{20} and A_{40} were initially zero; in Run 2, A_{20}, A_{40} and A_{22} were all originally zero; in Run 3, A_{20}, A_{40} and A_{11} were all originally zero; in Run 4, A_{40} was originally zero. One point to notice about Run 3 is that since A_{11} and A_{20} were both initially zero, they remain zero: those two modes cannot be generated by the interaction of the other two modes. It is interesting to note in these figures that the behaviour of one mode depends upon which other modes is present, which confirms that there is indeed a strong nonlinear interaction between the modes as suggested by [8, 9].

Conclusions

In the preceding sections, we have outlined the theory presented in [8, 9] for nonlinear interactions in the (plane)

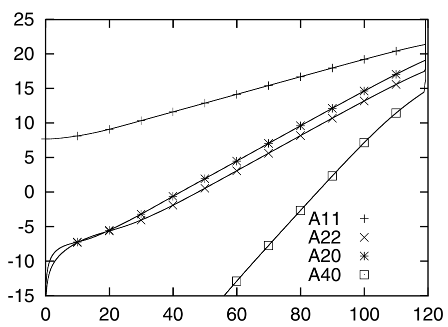

Figure 3: 3D Run2

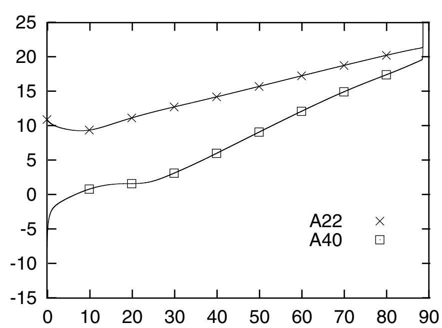

Figure 4: 3D Run3

Bickley jet between the varicose and sinuous modes, and then presented some of the three-dimensional results obtained using DNS. For both the three-dimensional case and also the two-dimensional case [7], the nonlinear theory suggests that there is an interaction between the sinuous modes and the varicose modes. The three-dimensional simulations presented here would appear to confirm that because the behaviour of a mode clearly differs depending upon which other modes are present. This behaviour is quite strong in the three-dimensional case and two-dimensional simulations not presented here indicate that the two-dimensional modes also interact but that the interaction in that case is much weaker. We are currently performing more simulations, although this is a slow process (the turnaround time for 3D runs on the UWO Cray is about 2 months), and we hope to present more complete results at some point in the future.

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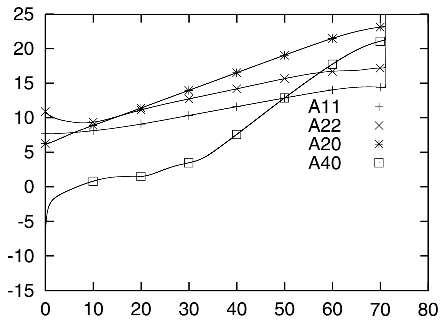

Figure 5: 3D Run4

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